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Game Logic and its Applications I*

Abstract. This paper provides a logic framework for investigations of game theoretical problems. We adopt an infinitary extension of classical predicate logic as the base logic of the framework. The reason for an infinitary extension is to express the common knowledge concept explicitly. Depending upon the choice of axioms on the knowledge operators, there is a hierarchy of logics. The limit case is an infinitary predicate extension of modal propositional logic *KD4*, and is of special interest in applications. In Part I, we develop the basic framework, and show some applications: an epistemic axiomatization of Nash equilibrium and formal undecidability on the playability of a game. To show the formal undecidability, we use a term existence theorem, which will be proved in Part II.

Key words: infinitary predicate *KD4*, common knowledge, Nash equilibrium, undecidability on playability.

1. Introduction

In the early stage of their literatures, game theory and mathematical logic had some common contributors, e.g., Zermelo, von Neumann and McKinsey, and then these fields had been developed with almost no interactions. Recently, the recognition of a possible relationship in aims and objects between them has been reemerging. The relationship may be summarized as the view that game theory is a theory of human behavior in social situations, while mathematical logic is a theory of mathematical practices by human beings. When we emphasize rational behavior in game theory, the relationship is even closer. In this paper, we take this view and develop a logic framework for investigations of game theory.

The primary purpose of the new framework is to understand the players' rational decision makings and their interactions in a game situation. In a game situation, each rational player thinks about his strategy choice, and there he may need to know and think about the other players' strategy choices, too, since their decisions affect those players interactively. Of course,

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some logical and introspective abilities are required for such thinking. Here epistemic aspects such as knowledge, logical and introspective abilities are entangled in the players' decision makings. We would like to develop our framework to encompass these features or some important part of them.

With respect to the feature of logical reasoning, we can find some literature called "epistemic logic" initiated by Hintikka [8]. Recently, epistemic logic is applied to the considerations of some game theoretical problems (cf., Bacharach [3] for a recent bibliography). Nevertheless, epistemic logic has been developed primarily in propositional logic. In game theory, the use of the real number system is standard, for example, the classical existence theorem of a Nash equilibrium in mixed strategies is proved in the real number system (von Neumann [23, 24] and Nash [20]). Hence we need to extend epistemic logic to predicate logic so as to formulate some real number theory.

Another important feature is the common knowledge concept. For the decision making of each player in a game situation, he may need to know the other players' knowledge and thinking about the situation. These knowledge and thinking may have a nested structure, e.g., he knows that the others know that he knows the game situation, and so on. This nested structure may form an infinite hierarchy, which is the problem of common knowledge. Common knowledge on the basic description of a game as well as on the logical and introspective abilities of the players may be required.

In the literature of epistemic logic, "fixed point logic" is developed to incorporate the common knowledge concept into finitary epistemic logic (cf., Halpern-Moses [6] and Lismont-Mongin [16]). There common knowledge is treated as a part of logic. Since common knowledge is an infinitary concept, we choose a framework in which infinitary conjunctions and disjunctions are allowed to express common knowledge explicitly as a logical formula, which enables us to treat common knowledge as an object of our logic instead of a part of our logic.[1] By choosing this research strategy, we can separate the development of the logical framework from its application to a particular game theoretical problem.

As a consequence of the above desiderata, the base logic, GL_0, of our framework is an infinitary extension of classical predicate logic. In this base logic, we formulate the logical abilities of the players as well as the knowledge of a game situation. The base logic may be regarded as the description of the logical ability of the outside investigator. We give essentially the same logical ability to each player, which is described inside the base logic. This

[1]Kaneko-Nagashima [12] argued in a proof theoretic manner that in a finitary logic without adding any inference rule on the common knowledge operator, it would be impossible to define the common knowledge concept. Segerberg [22] reached also a similar conclusion in a semantical manner.

is logic GL_p.

The next step is to give the introspective ability to each player. We assume that each knows what he knows, described by $K_i(A) \supset K_iK_i(A)$, and also that he knows his logical and introspective abilities. By these assumptions, we obtain logic GL_1. When there is only one player, logic GL_1 coincides with the infinitary predicate extension of modal propositional logic *KD4*.

When there are at least two players, logic GL_1 is much weaker than the extension of modal *KD4*. Here the knowledge of players about the other players' logical and introspective abilities are necessary to introduce. We have a hierarchy of logics

$$GL_0, GL_1, GL_2, ...,; \text{ and the limit } GL_\omega$$

by assuming that player i_1 knows that player i_2 knows ... player i_m knows the logical and introspective abilities of the players to various degrees from $m = 0$ to ω. When there are at least two players, the limit GL_ω coincides with the extension of modal *KD4*. For this equivalence, we need the common knowledge of the logical and introspective abilities of the players. Sections 2–4 are devoted for the development of these logics.

In Sections 5 and 6, we show possible applications of our framework to game theory. The first is an epistemic axiomatization of the Nash equilibrium concept. The axiomatization includes one epistemic aspect, which leads to the common knowledge of Nash equilibrium, $C(Nash_{\mathrm{g}}(\overrightarrow{a}))$, instead of Nash equilibrium itself. This axiomatization is formulated in logic GL_ω within the ordered field language. The additional common knowledge operator requires us to reconsider the playability of a game and the existence problem of a Nash equilibrium, which is the subject of Section 6.

The existence theorem of a Nash equilibrium by von Neumann [23], [24] and Nash [20] holds in the real closed field theory. It follows from this that the common knowledge of the existence of a Nash equilibrium, $C(\exists \overrightarrow{x} Nash_{\mathrm{g}}(\overrightarrow{x}))$, is derived from the common knowledge of the real closed field axioms. However, the axiomatization of Section 5 states that the existence quantifier must be outside the scope of the common knowledge, $\exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$, in order to have the playability of a game g, which is deductively stronger than $C(\exists \overrightarrow{x} Nash_{\mathrm{g}}(\overrightarrow{x}))$. In Section 6, we prove that the playability is formally undecidable for some three-person game g with a unique Nash equilibrium, that is, neither $\exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$ nor $\neg\exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$ is provable from the common knowledge of the real closed field axioms in logic GL_ω. Although this undecidability result is dependent upon the choice of a language and can be resolved by extending the language, it is the point that the players cannot realize the necessity of such an extension, since they know neither positive nor negative statement.

In Part II, we will develop sequent calculi of our logics in the Genzten style, and prove the cut-elimination theorem for them. The key theorem for the formal undecidability result of Section 6 of Part I will be proved, using the cut-elimination theorem.

2. Logics GL_0, GL_p and GL_1

2.1 Base logic GL_0

We adopt an infinitary language, based on the following list of symbols:

Free variables: $\mathbf{a}_0, \mathbf{a}_1, ...$; *Bound variables*: $\mathbf{x}_0, \mathbf{x}_1, ...$;

Functions: $f_0, f_1, ...$; *Predicates*: $P_0, P_1, ...$;

Knowledge operators: $K_1, ..., K_n$;

Logical connectives: $\neg$ (*not*), $\supset$ (*implies*), $\wedge$ (*and*), $\vee$ (*or*), $\forall$ (*for all*), $\exists$ (*exists*), where $\wedge$ and $\vee$ are allowed to be applied to infinitely many formulae;

Parentheses: (,).

The numbers of functions and predicates are arbitrary, except that there is at least one predicate. A 0-ary function is an individual constant, and a 0-ary predicate is a propositional variable. By the expression $K_i(A)$, we mean that player i knows that A is true.

The space of *terms* is defined by the standard finitary induction: (i) each free variable is a term; and (ii) if f_k is an ℓ–ary function and if $t_1, ..., t_\ell$ are terms, then $f_k(t_1, ..., t_\ell)$ is a term.

Let $\mathcal{P}_0$ be the set of all formulae generated by the standard finitary inductive definition with respect to $\neg, \supset, \forall, \exists$ and $K_1, ..., K_n$ from the atomic formulae. Suppose that $\mathcal{P}_t$ is already defined ($t = 0, 1, ...$). We call a non-empty countable subset Φ of $\mathcal{P}_t$ an *allowable set* iff it contains a finite number of free variables.[2] For an allowable set Φ, the expressions $(\wedge\Phi)$ and $(\vee\Phi)$ are considered here. From the union $\mathcal{P}_t \cup \{(\wedge\Phi), (\vee\Phi) : \Phi$ is an allowable set in $\mathcal{P}_t\}$, we obtain the space $\mathcal{P}_{t+1}$ of formulae by the standard finitary inductive definition with respect to $\neg, \supset, \forall, \exists$ and $K_1, ..., K_n$. We denote $\bigcup_{t<\omega} \mathcal{P}_t$ by $\mathcal{P}_\omega$.[3,4] An expression in $\mathcal{P}_\omega$ is called a *formula*. We abbreviate $\wedge\{A, B\}$ and $\vee\{A, B\}$ as $A \wedge B$ and $A \vee B$.

[2] This requirement will be used in Part II.

[3] Note that $\wedge\Phi$ and $\vee\Phi$ may not be in $\mathcal{P}_\omega$ for some countable subsets Φ of $\mathcal{P}_\omega$. For our purpose, however, this does not matter and the space $\mathcal{P}_\omega$ is large enough.

[4] This space is already uncountable. Some smaller, countable, space of formulae suffices

The primary reason for the infinitary language is to express common knowledge explicitly as a conjunctive formula. The common knowledge of a formula A is defined as follows: For any $m \geq 0$, we denote the set $\{K_{i_1}K_{i_2}...K_{i_m}$: each K_{i_t} is one of $K_1, ..., K_n$ and $i_t \neq i_{t+1}$ for all $t = 1, ..., m-1\}$ by $\mathsf{K}(m)$.[5] When $m = 0$, $K_{i_1}K_{i_2}...K_{i_m}$ is interpreted as the null symbol. We define the common knowledge of A by

$$\wedge\{K(A) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\},$$

which we denote by $C(A)$. If A is in $\mathcal{P}_m$, then $C(A)$ is in $\mathcal{P}_{m+1}$. Hence the space $\mathcal{P}_\omega$ is closed with respect to the operation $C(\cdot)$.

Base logic GL_0 is defined by the following seven axiom schemata and five inference rules: For any formulae A, B, C, allowable set Φ, and term t,

($L1$): $A \supset (B \supset A)$;

($L2$): $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

($L3$): $(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$;

($L4$): $\wedge\Phi \supset A$, where $A \in \Phi$;

($L5$): $A \supset \vee\Phi$, where $A \in \Phi$;

($L6$): $\forall x A(x) \supset A(t)$;

($L7$): $A(t) \supset \exists x A(x)$;

$$\frac{A \supset B \quad A}{B} \ (MP)$$

$$\frac{\{A \supset B : B \in \Phi\}}{A \supset \wedge\Phi} \ (\wedge\text{-}Rule) \qquad \frac{\{A \supset B : A \in \Phi\}}{\vee\Phi \supset B} \ (\vee\text{-}Rule)$$

$$\frac{A \supset B(a)}{A \supset \forall x B(x)} \ (\forall\text{-}Rule) \qquad \frac{A(a) \supset B}{\exists x A(x) \supset B} \ (\exists\text{-}Rule),$$

where the free variable a must not occur in $A \supset \forall x B(x)$ of ($\forall$-*Rule*) and $\exists x A(x) \supset B$ of ($\exists$-*Rule*).

Let Φ be an empty or allowable set and A a formula. A *proof* of A from Φ is a countable tree with the following properties: (i) every path from the

for our purpose. For example, a countable and constructive space of formulae is provided in Kaneko-Nagashima [11]. We adopt the above space for presentational simplicity.

[5]The requirement $i_t \neq i_{t+1}$ for all $t = 1, ..., m-1$ will be used in Part II.

root is finite; (ii) a formula is associated with each node, and the formula associated with each leaf is an instance of $(L1)$ – $(L7)$ or a formula in Φ; (iii) adjoining nodes together with their associated formulae form an instance of the above inferences; and (iv) all formulae occuring in P are in $\mathcal{P}_t$ for some $t < \omega$.[6] For any subset Γ of $\mathcal{P}_\omega$, a formula A is *provable from* Γ, denoted by $\Gamma \vdash_0 A$, iff there is an empty or allowable subset Φ of Γ and a proof of A from Φ.

Logic GL_0 is an infinitary extension of finitary classical predicate logic. Hence we can freely use provable finitary formulae in classical logic. In fact, it is sound and complete with respect to the standard interpretation with infinitary conjunctions and disjunctions. That is, all valid formulae in this sense are provable, and *vice versa.* The following are some examples of provable formulae:

(1): $\vdash_0 A \vee B \equiv (\neg A \supset B)$;

(2): $\vdash_0 A \wedge B \equiv \neg(A \supset \neg B)$;

(3): $\vdash_0 (A \supset (B \supset C)) \equiv (A \wedge B \supset C)$;

(4): $\vdash_0 \neg \bigvee \Phi \equiv \bigwedge\{\neg A : A \in \Phi\}$, where Φ is an allowable set.

Here $A \equiv B$ denotes $(A \supset B) \wedge (B \supset A)$. We will not refer to those basic results in the following sections. We just mention the deduction theorem for the purpose of comparisons with modal logic. The above formula (3) is needed to prove this lemma.[7]

LEMMA 2.1. *[Deduction Theorem] Let A be a closed formula. If* $\Gamma \cup \{A\} \vdash_0 B$, *then* $\Gamma \vdash_0 A \supset B$.

Our base logic GL_0 can be regarded as a fragment of infinitary logic $L_{\omega_1\omega}$ (except the addition of multiple knowledge operator symbols) (cf., Karp [13] and Keisler [14]). As a space of formulae, $\mathcal{P}_\omega$, is much smaller than the space of formulae in $L_{\omega_1\omega}$.[8]. Since our primary purpose of the infinitary extension is to express common knowledge explicitly as a conjunctive formula, the present extension suffices for our purpose.

[6]In the following, we use the transfinite induction proof on the structure of a proof tree from the leaves to the root (or from the root to the leaves). That is, if the following two steps, (1) and (2), are proved for a property p, then p holds for all nodes: (1) the property p holds for all leaves; and (2) for any node x, if the property p holds for any node y immediately above x, then p holds also for x. This is derived from a weak form of Zorn's lemma.

[7]The proofs are available from the authors on request.

[8]We will evaluate the "depth" of a formula A, called the grade of A, in Part II. The grade of any formula in $\mathcal{P}_\omega$ is smaller than ordinal ω^2.

2.2 Logic GL_p: Players' logical abilities

Logic GL_0 may be regarded as a description of the logical ability of the outside theorist, whom we call the *investigator*. Besides this description, the investigator may have the set of assumptions Γ, in which the basic knowledge of each player is described, i.e., $\{K_i(A) : K_i(A) \in \Gamma\}$. Player i may deduce more knowledge from the basic knowledge, but unless he is given some logical ability, he could not derive any new knowledge from the basic knowledge. In this subsection, we will give each player essentially the same logical ability as the investigator's. That is, we define logic GL_p and prove that each player is given the same logical ability as the investigator's. Logic GL_p is the starting point of our game logic development. By adding some introspective axioms to the axioms for the description of the pure logical ability of each player, we will obtain logic GL_1 in the next subsection. We will mention some other choices of epistemic logics in the end of Section 3.

We assume that each player $i = 1, ..., n$ knows the logical axioms $L1$ – $L7$. For example, the knowledge of $L1$ is described as $K_i(A \supset (B \supset A))$, which is denoted by $L1_i$. Similarly, we define $L2_i$ – $L7_i$. We also assume that each player has the inference ability corresponding to *MP*, ($\wedge$-*Rule*), ($\vee$-*Rule*), ($\forall$-*Rule*), ($\exists$-*Rule*):

$$(MP_i): \quad K_i(A \supset B) \wedge K_i(A) \supset K_i(B);$$

$$(\wedge_i): \quad K_i(\wedge\{A \supset B : B \in \Phi\}) \supset K_i(A \supset \wedge\Phi);$$

$$(\vee_i): \quad K_i(\wedge\{A \supset B : A \in \Phi\}) \supset K_i(\vee\Phi \supset B);$$

$$(\forall_i): \quad K_i(\forall x(A \supset B(x))) \supset K_i(A \supset \forall x B(x));$$

$$(\exists_i): \quad K_i(\forall x(A(x) \supset B)) \supset K_i(\exists x A(x) \supset B),$$

where A, B are any formulae, Φ an allowable set, and x a bound variable.

The above schemata are reformulations of inference rules *MP* – ($\exists$-*Rule*). Here the investigator has the description of the logical ability of each player i, and can deduce what player i may deduce. This description is made in the object language, while the investigator's logical ability is described in the metalanguage.

For the connection between the investigator's and the players' knowledge, we make the minimum requirement:

$$(\bot_i): \quad \neg K_i(\neg A \wedge A),$$

where A is any formula and $i = 1, ..., n$. This requires that no contradiction be derived from player i's basic knowledge.

We add two more axioms, which we call the *Barcan axioms*:

($\wedge$-B_i): $\wedge K_i(\Phi) \supset K_i(\wedge\Phi)$;

($\forall$-B_i): $\forall x K_i(A(x)) \supset K_i(\forall x A(x))$;

where Φ is an allowable set and $K_i(\Phi)$ denotes the set $\{K_i(A) : A \in \Phi\}$. When Φ is finite, ($\wedge$-B_i) is derived from other axioms, but is needed for infinite Φ. Axiom ($\wedge$-B_i) will be used to derive the property:

$$C(A) \supset K_i(C(A)) \quad \text{for } i = 1, ..., n. \tag{2.1}$$

That is, if A is common knowledge, then each player i knows that it is common knowledge. This will be provable in GL_1 and play an important role in game theoretic applications.[9] Those two axioms are also needed to show the equvalence between the formulations of game logics in Part I and those in Part II.

Logic GL_p is defined by the sets of all instances of $L1_i$ – $L7_i$, (MP_i) – $(\exists_i)$, $(\bot_i)$, ($\wedge$-B_i) and ($\forall$-B_i), denoted by Δ_{ip}, for $i = 1, ..., n$. That is, for any set Γ of formulae and any formula A, we define the provability $\vdash_p$ in GL_p by

$$\Gamma \vdash_p A \quad \text{iff} \quad \Gamma \cup (\bigcup_i \Delta_{ip}) \vdash_0 A. \tag{2.2}$$

When $\Gamma \vdash_p A$, the investigator deduces A from Γ, using his knowledge of i's logical ability described by Δ_{ip} as well as using player i's knowledge described in Γ. When $K_i(\Gamma) \vdash_p K_i(A)$, the investigator deduces that player i deduces A from his basic knowledge $K_i(\Gamma)$. The following proposition states that each player is given the same logical ability as the investigator's.

PROPOSITION 2.2. *[Faithful Representation] Let Γ be a set of closed formulae. Then $K_i(\Gamma) \vdash_p K_i(A)$ if and only if $\Gamma \vdash_0 A$.*

The *if* part of Proposition 2.2 is proved by putting the outer K_i to each formula in a proof of A from Γ. Note that ($\wedge$-B_i) and ($\forall$-B_i) are needed here. The *only-if* part will be proved in Part II, using the cut-elimination theorem for GL_p.

Since GL_0 describing the logical ability of the investigator is sound and complete, the logical ability of each player is also complete in the sense of the infinitary extension of classical logic.

Provability $\vdash_p$ has the following properties.

[9] In GL_p, (2.1) does not necessarily hold. However, if we eliminate condition $i_t \neq i_{t+1}$ in the definition of $K(t)$, then (2.1) would hold in GL_p.

PROPOSITION 2.3. *Let A be a formula, Φ an allowable set of formulae, and x a bound variable. Then*

$(\wedge)$: $\vdash_p K_i(\wedge\Phi) \equiv \wedge K_i(\Phi)$;

$(\vee)$: $\vdash_p \vee K_i(\Phi) \supset K_i(\vee\Phi)$;

$(\forall)$: $\vdash_p K_i(\forall x A(x)) \equiv \forall x K_i(A(x))$;

$(\exists)$: $\vdash_p \exists x K_i(A(x)) \supset K_i(\exists x A(x))$.

PROOF. We prove only $(\forall)$. Since $\forall x K_i(A(x)) \supset K_i(\forall x A(x))$ is $(\forall\text{-}B_i)$, we have to prove the converse. Since $K_i(\forall x A(x) \supset A(a))$ is $L6_i$, we have, from (MP_i), $\vdash_p K_i(\forall x A(x)) \supset K_i(A(a))$, where the free variable a is taken so that it does not occur in $K_i(\forall x A(x))$. Hence $\vdash_p K_i(\forall x A(x)) \supset \forall x K_i(A(x))$ by $(\forall\text{-}Rule)$. ■

Notice the asymmetries between $(\wedge)$ and $(\vee)$ and between $(\forall)$ and $(\exists)$. Consider the first one. The direction $\supset$ in $(\wedge)$ is the dual of $(\vee)$, which is provable without $(\wedge\text{-}B_i)$, and the other direction is $(\wedge\text{-}B_i)$ itself. As was mentioned, $(\wedge\text{-}B_i)$ is necessary for game theoretical applications; without it, we could not have the crucial property (2.1), which does not hold yet in GL_p. On the other hand, if we had equivalence in $(\vee)$, then we would be incapable of considering some apparently important issues: If this was the case, for example, $\vdash_p K_i(\neg A \vee A)$ would be equivalent to $\vdash_p K_i(\neg A) \vee K_i(A)$. The first one always holds since player i has the logical ability without having further knowledge on A, but the second requires that player i know that A is true or $\neg A$ is true, which depends upon some knowledge specific to A. Therefore the asymmetry is needed for further developments. The parallel argument is applied to the asymmetry between $(\forall)$ and $(\exists)$.

2.3 Logic GL_1: Players' logical and introspective abilities

In logic GL_p, as was shown in Proposition 2.2, each player has the same logical ability as the investigator. Nevertheless, he may know neither his own logical ability nor that he knows something. For example, $K_1(K_1(A \supset B) \wedge K_1(A) \supset K_1(B))$ is not necessarily provable in GL_p. We define another logic GL_1 by adding introspective abilities of players. Introspective abilities consists of two parts: (i) if a player knows A, then he knows that he knows A; and (ii) he knows his logical and introspective abilities themselves. The addition of these introspective abilities to our framework is desirable for several reasons, which will be clear later.

Formally, the following, called the *Positive Introspection axiom*, describes (i):

(PI_i): $K_i(A) \supset K_iK_i(A)$,

where A is an arbitrary formula. The requirement (ii) is obtained by putting K_i to each formula in Δ_{ip} and of (PI_i). That is, we denote the union of Δ_{ip} and the set of all instances of (PI_i) $(i = 1, ..., n)$ by Δ_{i0}, and denote $\Delta_{i0} \cup \{K_i(A) : A \in \Delta_{i0}\}$ by Δ_{i1}. We define the *provability* $\vdash_1$ in GL_1 by

$$\Gamma \vdash_1 A \text{ iff } \Gamma \cup (\bigcup_i \Delta_{i1}) \vdash_0 A. \tag{2.3}$$

In this logic, (2.1) is provable, that is,

LEMMA 2.4. $\vdash_1 C(A) \supset K_i(C(A))$ *for any* $i = 1, ..., n$.

PROOF. It holds that $\vdash_1 C(A) \supset \wedge\{K_iK(A) : K_iK \in \bigcup_{t<\omega} \mathsf{K}(t)\}$. By $(\wedge\text{-}B_i)$, we have

$$\vdash_1 C(A) \supset K_i \left(\wedge\{K(A) : K_iK \in \bigcup_{t<\omega} \mathsf{K}(t)\} \right). \tag{2.4}$$

By (PI_i), $\vdash_1$ $C(A) \supset K_iK_i\,(\wedge\{K(A) : K_iK \in \bigcup_{t<\omega} \mathsf{K}(t)\})$. Since $\vdash_1$ $K_i(K_i(\wedge\Phi) \supset \wedge K_i(\Phi))$ for any allowable Φ, we have $\vdash_1$ $C(A) \supset K_i(\wedge \{K_iK(A) : K_iK \in \bigcup_{t<\omega} \mathsf{K}(t)\})$. This together with (2.4) implies $\vdash_1 C(A) \supset K_i(C(A))$. ■

As was stated, Lemma 2.4 is not necessarily proved without the Barcan axiom $(\wedge\text{-}B_i)$. This will be discussed briefly in Part II.

Logic GL_1 is of special interests, since it can be regarded as an infinitary predicate extension of modal logic *KD4* when there is only one player, i.e., $n = 1$. We define provability $\vdash_{KD4}$ from $\vdash_0$ by adding (MP_i), $(\wedge\text{-}B_i)$, $(\forall\text{-}B_i)$, $(\bot_i)$, (PI_i) and

$$\frac{A}{K_i(A)} \quad (Necessitation)$$

for $i = 1, ..., n$.[10]

PROPOSITION 2.5. *Let* $n = 1$. *Let* Φ *be an allowable set of closed formulae, and* A *a formula. Then* $\Phi \vdash_1 A$ *if and only if* $\vdash_{KD4} \wedge\Phi \supset A$.

[10]Axioms $(\wedge_i), (\vee_i), (\forall_i)$ and $(\exists_i)$ are derived in this extension.

When $n \geq 2$, this relationship breaks down, and provability $\vdash_1$ is much weaker than the corresponding $\vdash_{KD4}$. For example, $K_2K_1\,(A \supset (B \supset A))$ is not provable in GL_1. To have the equivalence between them, we need to assume that every formula in $\bigcup_i \Delta_{i1}$ is common knowledge among the players. This means that there is an infinite hierarchy from $\vdash_1$ to $\vdash_{KD4}$. This is the subject of Section 3.

Since $\Phi \vdash_1 A$ is equivalent to $\vdash_1 \wedge\Phi \supset A$, it suffices to show, for Proposition 2.5, that $\vdash_1 A$ is equivalent to $\vdash_{KD4} A$. The *only-if* part is straightforward, and the *if* part follows from Lemma 2.6.

LEMMA 2.6. *[Necessitation] Let $n = 1$. Then $\vdash_1 A$ implies $\vdash_1 K_1(A)$.*

PROOF. Suppose $\vdash_1 A$, i.e., $\Delta_{11} \vdash_0 A$. Then there is a proof of A from Δ_{11}. By induction on the tree structure from leaves, we prove $\Delta_{11} \vdash_0 K_1(B)$ for any formula B in the proof. Let B be an initial formula in the proof. Then B is an instance of $L1$ – $L7$ or is a formula in Δ_{11}. If B is expressed as $K_1(B')$, then $\Delta_{11} \vdash_0 K_1K_1(B')$ by (PI_1), and otherwise, $K_1(B)$ is in Δ_{11}, so $\Delta_{11} \vdash_0 K_1(B)$. Assume the induction hypothesis that $\Delta_{11} \vdash_0 K_1(C)$ for any immediate predecessor C of an occurrence of B in the proof. Then we have to consider the five inference rules. Here we consider only ($\forall$-*Rule*). Then B takes the form $D \supset \forall x E(x)$ and the unique immediate predecessor C takes the form $D \supset E(a)$, where a does not occur in and $D \supset \forall x E(x)$. By the induction hypothesis, $\Delta_{11} \vdash_0 K_1(D \supset E(a))$. Then $\Delta_{11} \vdash_0 \forall x K_1(D \supset E(x))$, which together with ($\forall$-$B_i$) implies $\Delta_{11} \vdash_0 K_1(\forall x(D \supset E(x))$. Thus we have $\Delta_{11} \vdash_0 K_1(D \supset \forall x E(x))$ by $(\forall_i)$. ■

3. Iterated knowledge of deductive abilities

In logic GL_1 with at least two players, each player does not know the other players' logical and introspective abilities, though he has and knows his own. Once a player knows their abilities, it would be possible for him to infer what the others deductively know. This knowledge of players' logical and introspective abilities may have a nested structure, for example, player i_1 knows that player i_2 knows ... i_m knows those abilities. Thus there is an infinite hierarchy of logics with the various degrees of nestedness. When there are at least two players, only the limit GL_ω coincides with the infinitary predicate extension of modal propositional logic *KD4*. This limit case is particularly important for our applications to game theory in Sections 5 and 6.

3.1 Game logics GL_m $(0 \leq m \leq \omega)$

The idea that a player knows his and the others' logical and introspective abilities is described by assuming that every formula in $\bigcup_i \Delta_{i1}$ is known to the players in the nested manner. Define Δ_m for any $m \leq \omega$ by

$$\Delta_m = \{K(A) : A \in \bigcup_i \Delta_{i1} \text{ and } K \in \bigcup_{t<m} \mathsf{K}(t)\}. \tag{3.5}$$

Recall that $\mathsf{K}(t)$ is the set $\{K_{i_1}K_{i_2}...K_{i_t}$: each K_{i_k} is one of $K_1, ..., K_n$ and $i_k \neq i_{k+1}$ for all $k = 1, ..., t-1\}$. Let Γ be a set of formulae. Then we define the provability $\vdash_m$ in logic GL_m by

$$\Gamma \vdash_m A \ \text{ iff } \Gamma \cup \Delta_m \vdash_0 A. \tag{3.6}$$

Of course, $m < k$ and $\Gamma \vdash_m A$ imply $\Gamma \vdash_k A$.

In logic GL_m $(m < \omega)$, the players know the logical and introspective abilities of the players up to the depth m in the sense that player i_1 knows that player i_2 knows ... that player i_m knows those abilities. In GL_ω, the players know the abilities up to any depth. That is, the abilities of players are common knowledge among the players.

First, we give some lists of provable formulae in GL_m.

PROPOSITION 3.1. *For any m with $1 \leq m \leq \omega$ and any $L \in \{KK_i : K \in \bigcup_{t<m} \mathsf{K}(t)$ and $i = 1, ..., n\}$,*

(MP_L): $\vdash_m L(A \supset B) \wedge L(A) \supset L(B)$;

$(\wedge_L)$: $\vdash_m L(\wedge\{A \supset B : B \in \Phi\}) \supset L(A \supset \wedge\Phi)$;

$(\vee_L)$: $\vdash_m L(\wedge\{A \supset B : A \in \Phi\} \supset L(\vee\Phi \supset B)$;

$(\forall_L)$: $\vdash_m L(\forall x(A \supset B(x))) \supset L(A \supset \forall x B(x))$;

$(\exists_L)$: $\vdash_m L(\forall x(A(x) \supset B)) \supset L(\exists x A(x) \supset B)$;

$(\perp_L)$: $\vdash_m \neg L(\neg A \wedge A)$;

$(\wedge\text{-}B_L)$: $\vdash_m \wedge L(\Phi) \supset L(\wedge\Phi)$;

$(\forall\text{-}B_L)$: $\vdash_m \forall x L(A(x)) \supset L(\forall x A(x))$;

(PI_L): $\vdash_m L(A) \supset LK_i(A)$,

where A, B are formulae, Φ an allowable set, $L(\Phi)$ the set $\{L(C) : C \in \Phi\}$, and x a bound variable.

PROOF. When L is K_i, (MP_L) – (PI_L) are axioms (MP_i) – (PI_i) and belong to $\bigcup_i \Delta_{i1}$. Assume the induction hypothesis that (MP_L) – (PI_L) hold in GL_m for any $L \in \{KK_i : K \in \mathsf{K}(\ell)$ and $i = 1, ..., n\}$, where $\ell + 2 \leq m$ if $m < \omega$ and $\ell < \omega$ if $m = \omega$.

Let $K \in \mathsf{K}(\ell)$. Since $K(K_i(A \supset B) \wedge K_i(A) \supset K_i(B)) \in \Delta_m$, we have, using (MP_L), $\vdash_m KK_i(A \supset B) \wedge KK_i(A) \supset KK_i(B)$. The other assertions can be proved in the same manner. ■

Note that in GL_ω, (MP_L) – $(\forall\text{-}B_L)$ hold for the common knowledge operator $C(\cdot)$ in the replacement of $L(\cdot)$. Assertion (PI_L) is changed into $C(K_i(A)) \supset C(K_iK_i(A))$.

Observe that the claims of this proposition are parallel to the axioms, MP_i – (PI_i) with the replacement of K_i by L. The formulae corresponding to $L1_i$ – $L7_i$, e.g., $L1_L : L(A \supset (B \supset A))$, belong to Δ_m by (3.5). Hence, by substituting L for K_i in the assertions of Proposition 2.3, we have the following.

PROPOSITION 3.2. *For any m with $1 \leq m \leq \omega$ and any $L \in \{KK_i : K \in \bigcup_{t<m} \mathsf{K}(t)$ and $i = 1, ..., n\}$,*

$(\wedge_L)$: $\vdash_m L(\wedge\Phi) \equiv \wedge L(\Phi)$;

$(\vee_L)$: $\vdash_m \vee L(\Phi) \supset L(\vee\Phi)$;

$(\forall_L)$: $\vdash_m L(\forall x A(x)) \equiv \forall x L(A(x))$;

$(\exists_L)$: $\vdash_m \exists x L(A(x)) \supset L(\exists x A(x))$,

where A, B are formulae, Φ an allowable set, and x a bound variable.

Thus the same asymmetries as in Proposition 2.3 appear in GL_m. These asymmetries remain for the common knowledge formula, that is,

$(\wedge_C)$: $\vdash_\omega C(\wedge\Phi) \equiv \wedge C(\Phi)$;

$(\vee_C)$: $\vdash_\omega \vee C(\Phi) \supset C(\vee\Phi)$;

$(\forall_C)$: $\vdash_\omega C(\forall x A(x)) \equiv \forall x C(A(x))$;

$(\exists_C)$: $\vdash_\omega \exists x C(A(x)) \supset C(\exists x A(x))$,

where $C(\Phi)$ is the set $\{C(B) : B \in \Phi\}$. Especially, $(\exists_C)$ plays an important role in Section 6.

The following properties hold on common knowledge.

PROPOSITION 3.3. *Let Γ be a set of formulae, and A a formula. Then*

1) *(Necessitation):* $C(\Gamma) \vdash_\omega A$ *imply* $C(\Gamma) \vdash_\omega K_i(A)$;

2): $\Gamma \vdash_0 A$ *implies* $C(\Gamma) \vdash_\omega C(A)$;

3): $C(\Gamma) \vdash_\omega A$ *if and only if* $C(\Gamma) \vdash_\omega C(A)$.

PROOF. 1): This can be proved by induction on the proof of A from $C(\Gamma)$ in GL_ω. The only crucial step is to show that for any initial formula B in the proof, we have $\vdash_\omega K_i(B)$ if $B \in \Delta_\omega$ or $C(\Gamma) \vdash_\omega K_i(B)$ if $B \in C(\Gamma)$. The first follows from (PI_i). In the second case, Lemma 2.4 is used.
2): If $\Gamma \vdash_0 A$, then it can be proved using Proposition 3.2 that $K(\Gamma) \vdash_\omega K(A)$ for any $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. Hence $C(\Gamma) \vdash_\omega K(A)$ for any K. Thus $C(\Gamma) \vdash_\omega C(A)$ by ($\wedge$-*Rule*).
3): By (1), $C(\Gamma) \vdash_\omega A$ implies $C(\Gamma) \vdash_\omega K(A)$ for any $K \in \bigcup_{m<\omega} \mathsf{K}(m)$. Thus $C(\Gamma) \vdash_\omega C(A)$. The converse is straightforward.

Note that in the above three proofs, we used ($\forall$-B_i) as well as ($\wedge$-B_i). ∎

3.2 Relationship to modal logic

As was already mentioned, when $n \geq 2$, we need to go to the limit GL_ω to make a direct comparison to modal logic *KD4*.

PROPOSITION 3.4. *Let $n \geq 2$. Let Φ be an allowable set of closed formulae, and A a formula. Then $\Phi \vdash_\omega A$ if and only if $\vdash_{KD4} \wedge\Phi \supset A$.*

PROOF. It suffices to prove that $\vdash_\omega A$ if and only if $\vdash_{KD4} A$. The *only-if* part is straightforward. We can prove the *if* part using Proposition 3.3.1). ∎

Thus when we assume the common knowledge of the logical and introspective abilities of the players, our logic, GL_ω, becomes equivalent to the infinitary predicate extension of *KD4*.

Proposition 3.4 as well as Proposition 2.5 hold in the finitary fragment of our framework. Hence these results are not dependent upon the infinitary extension. The reason for this fact is that the common knowledge assumption of logical abilities are not needed to be described as object formulae for these equivalences. The infinitary extension plays an essential role when we discuss common knowledge as an object formula in the logic such as (2.1).

3.3 Logic GL_{mp} $(1 \leq m \leq \omega)$

We can obtain another hierarchy of logics based on $\bigcup_i \Delta_{ip}$ instead of $\bigcup_i \Delta_{i1}$ also with the substitution of $\mathsf{K}_p(t) = \{K_{i_1}K_{i_2}...K_{i_t}$: each K_{i_t} is one of $K_1, ..., K_n\}$ for $\mathsf{K}(t)$ for each t. That is, the provability $\vdash_{mp}$ in GL_{mp} is defined by

$$\Gamma \vdash_{mp} A \text{ iff } \Gamma \cup \Delta_{mp} \vdash A, \tag{3.7}$$

where $\Delta_{mp} = \{K(A) : A \in \bigcup_i \Delta_{ip} \text{ and } K \in \bigcup_{t<m} \mathsf{K}_p(t)\}$. When $m = 1$, $\vdash_{mp}$ is $\vdash_p$.

The limit logic $GL_{\omega p}$ becomes equivalent to the infinitary predicate extension of modal KD for any $n \geq 1$. Even for $n = 1$, since GL_{mp} does not allow the Positive Introspection axiom, we need to go to the limit logic $GL_{\omega p}$ to have the Necessitation property. For the same reason, we need $\mathsf{K}_p(t)$ instead of $\mathsf{K}(t)$ to obtain the repetition of K_i. To obtain the exact form of Proposition 3.3.1) for $GL_{\omega p}$, we need to modify the definition of common knowledge using $\mathsf{K}_p(t)$ instead of $\mathsf{K}(t)$ $(t \geq 0)$.

Similarly, we have different hierarchies when we choose some other axioms. When we eliminate $(\bot_i)$ from Δ_{ip} in the definition of logic $GL_{\omega p}$, the logic is an extension of modal logic K. Conversely, if we add, to Δ_{ip}, the first of or both of

(*Veridicality Axiom*): $K_i(A) \supset A$;

(*Negative Introspection Axiom*): $\neg K_i(A) \supset K_i(\neg K_i(A))$.

then the limit logics are equivalent to the infinitary predicate extensions of modal logic *S4* and *S5*, respectively. The extension $GL_{\omega S4}$ corresponding to *S4* is of special interest; it is convenient for some purpose and is an extension of the epistemic logic often discussed.[11]

Thus we can have a lot of variations of our logic GL_m. The reason of our choice of GL_m is philosophical as well as practical. Philosophically, GL_m can allow cognitive relativism in that only consistency in each player's knowledge is required, while allowing the logical and introspective abilities of the players. The Veridicality does not allow this cognitive relativism in that "truth" would be defined from the viewpoint of the investigator. Also, the Negative Introspection axiom requires metaknowledge. Conversely, the

[11] In the literature of epistemic logic, "knowledge" is sometimes distinguished from "belief" by adding the Veridicality Axiom. Hence if we follow this convention, our K_i should be called the *belief* operator, and the *common belief* of A is formulated as $\bigwedge\{K(A) : K \in \bigcup_{0<m<\omega} \mathsf{K}(m)\}$.

logical axioms as well as the Positive Introspection axiom seem to be natural requirements for players.

Practically, although a stronger one is often convenient for game theoretical applications, GL_m has better proof theoretic properties than stronger ones in that the cut-elimination theorem which will be used to evaluate provability would be more powerful in GL_m than in other stronger logics. This fact can be also interpreted from the fact that GL_m permits cognitive relativism, which will be briefly discussed in Part II. Specifically, the undecidability results of Section 6 will not be available in the *S4-type* extension.

In Sections 5 and 6, we use the limit logic GL_ω. The reason for this is that the infinite regress derived from a game theoretical consideration can be solved only in GL_ω.

4. Conservativeness of GL_m $(1 \leq m \leq \omega)$

This section shows that for any m $(1 \leq m \leq \omega)$, logic GL_m is a conservative extension of the infinitary extension GL_0 of classical logic. This conservativeness will play important roles in many ways.

A formula A is said to be *nonepistemic* iff it does not contain any $K_1,...,K_n$. Let ϵA be the formula obtained from A by eliminating all $K_1,...,K_n$, which is, more precisely, defined by induction on the structure of a formula. Of course, ϵA is nonepistemic. We denote $\{\epsilon A : A \in \Phi\}$ by $\epsilon\Phi$. Observing that any formula in $\epsilon\Delta_m$ is provable in GL_0, for example, $\epsilon(K_i(A \supset B) \wedge K_i(A) \supset K_i(B))$ is $(\epsilon A \supset \epsilon B) \wedge \epsilon A \supset \epsilon B$, we have the following proposition.

PROPOSITION 4.1. *[Conservative Extension] Let* Γ *be a subset of* $\mathcal{P}_\omega$ *and* A *a formula in* $\mathcal{P}_\omega$. *Then* $\Gamma \vdash_m A$ *implies* $\epsilon\Gamma \vdash_0 \epsilon A$.

The next proposition follows immediately from Proposition 4.1, which implies that the consistency of GL_m is reduced into that of GL_0. The consistency of GL_0 can be proved in the standard (semantic) manner.

PROPOSITION 4.2. *[Relative Consistency] Let* Γ *be a subset of* $\mathcal{P}_\omega$. *If* $\epsilon\Gamma$ *is consistent with respect to* $\vdash_0$, *then* Γ *is consistent with respect to* $\vdash_m$.

The following fact will be important in Section 6: Let Γ be a set of nonepistemic formulae and A a nonepistemic formula. Then

$$\begin{array}{c} C(\Gamma) \vdash_\omega \neg\exists x_1...\exists x_\ell C(A(x_1,...,x_\ell)) \\ \text{if and only if} \\ C(\Gamma) \vdash_\omega C(\neg\exists x_1...\exists x_\ell A(x_1,...,x_\ell)). \end{array} \tag{4.8}$$

In contrast with $(\exists_C)$ of Section 3, there is no distinction between these two negative existential statements. The above equivalence is proved as follows: The *only-if* part: by Proposition 4.1, we have, from the former, $\Gamma \vdash_0 \neg\exists x_1...\exists x_\ell A(x_1, ..., x_\ell)$, which together with Proposition 3.3.2) implies $C(\Gamma) \vdash_\omega C(\neg\exists x_1...\exists x_\ell A(x_1, ..., x_\ell))$. The *if* part: Since $\vdash_\omega C(\neg\exists x_1... \exists x_\ell A(x_1, ..., x_\ell)) \supset \neg\exists x_1...\exists x_\ell A(x_1, ..., x_\ell)$ and $\vdash_\omega \neg\exists x_1...\exists x_\ell A(x_1, ..., x_\ell) \supset \neg C(\exists x_1...\exists x_\ell A(x_1, ..., x_\ell))$, we have $\vdash_\omega C(\neg\exists x_1...\exists x_\ell A(x_1, ..., x_\ell)) \supset \neg C(\exists x_1... \exists x_\ell A(x_1, ..., x_\ell))$. Since $\vdash_\omega \neg C(\exists x_1...\exists x_\ell A(x_1, ..., x_\ell)) \supset \neg\exists x_1...\exists x_\ell C(A(x_1, ..., x_\ell))$ by $(\exists_C)$ of Section 3, we have $\vdash_\omega C(\neg\exists x_1...\exists x_\ell A(x_1, ..., x_\ell)) \supset \neg\exists x_1... \exists x_\ell C(A(x_1, ..., x_\ell))$.

5. Applications to Game Theory I: Epistemic axiomatization of Nash Equilibrium

This and following sections provide applications of our framework to game theory. Since classical game theory is described in the real number system, we need to specify a language and axioms for a real number theory. We use the standard language and axioms for the ordered field theory in this section, and will use the real closed field theory in Section 6. These are sufficient for the consideration of classical game theory. This section gives an epistemic axiomatization of Nash equilibrium, based on Kaneko-Nagashima [11] and Kaneko [9].[12] The result of the axiomatization deviates slightly from Nash equilibrium in classical game theory in that it becomes the common knowledge of Nash equilibrium. This can be regarded rather as faithful to the intended interpretation of Nash equilibrium in game theory. But this additional common knowledge operator requires us to reconsider a deeper problem of the playability of a game, which will be the subject of Section 6.

In these two sections, we use game logic GL_ω. The consideration of the present section cannot be done in logic GL_m for finite m. The ordered field axioms are, in fact, not used in this section, but the ordered field language suffices for the present purpose. For the existence problem of a Nash equilibrium, those axioms are needed.

[12] We can find some axiomatizations of Nash equilibrium in the recent game theoretical literature: Bacharach [2] made some axiomatic requirements for individual decision making in a game situation, and proved that such requirements are inconsistent even for a game with a unique Nash equilibrium. Aumann [1] gave an epistemic consideration of Nash equilibrium in a game with perfect information. Balkenborg-Winter [4] showed that common knowledge is not necessary in the case of a game with perfect information. This should be compared with our epistemic axiomatization. For other related game theoretical problems, see Kaneko-Nagashima [11] and Kaneko [9].

5.1 Language and basic game theoretic concepts

Here we specify the list of basic symbols:

Constants: **0**, **1**; *Binary functions*: $+, -, \cdot, /$;
Binary predicates: $\geq, =$; and *ℓ-ary predicates*: $D_1, ..., D_n$,

in addition to the other basic symbols specified in Section 2. The *ℓ-ary* predicates $D_1, ..., D_n$ are prepared for the epistemic consideration of Nash equilibrium. The other symbols are prepared for the description of the ordered field theory. We denote the set of all *ordered field axioms* and *equality axioms* by Φ_{of} (cf., Mendelson [18], [19]). We use the same symbol = for formal and informal equalities, which should not cause confusions.

First, we describe a noncooperative game in *informal mathematics.* Consider an *n-person finite* game **g**. For simplicity, we assume that each player has the same finite number, ℓ, of pure strategies. The payoff to player i from a *pure strategy combination* $(s_1, ..., s_n)$ is given as a rational number $\mathbf{g}_i(s_1, ..., s_n)$. The two-person game of Table 1 is called the "Prisoner's dilemma", where each player $i = 1, 2$ has two pure strategies N (*not confess*) and C (*confess*). Each vector in the table is a pair of payoffs to the players, e.g., $(\mathbf{g}_1(N, C), \mathbf{g}_2(N, C)) = (1, 6)$. We allow also mixed strategies, where a mixed strategy for player i is a probability distribution over his pure strategies.

	N	C
N	$(5, 5)$	$(1, 6)$
C	$(6, 1)$	$(2, 2)$

Table 1

	B	M
B	$(2, 1)$	$(0, 0)$
M	$(0, 0)$	$(1, 2)$

Table 2[13]

Now we formulate those game theoretical concepts in our formal language. First, we define *numerals* as follows: [0] is **0**, $[m]$ is $[m-1] + \mathbf{1}$ for an positive integer m, and $[m]$ is $\mathbf{0} - [-m]$ for a negative integer m. For a rational number $q = m/k$ (m/k are irreducible and $k > 1$), we define $[q]$ to be $[m]/[k]$. Thus numerals are closed terms.

Using numerals, the above game **g** is described in our language as follows: the payoff to player i from a strategy combination $(s_1, ..., s_n)$ is given as $[\mathbf{g}_i(s_1, ..., s_n)]$. A *mixed strategy* for player i is a vector of free variables $\overrightarrow{a}_i = (a_{i1}, ..., a_{i\ell})$ satisfying the following formula:

$$\left(\sum_{t=1}^{\ell} a_{it} = \mathbf{1}\right) \wedge \left(\bigwedge\{a_{it} \geq \mathbf{0} : t = 1, ..., \ell\}\right), \tag{5.9}$$

[13]See Luce-Raiffa [17] for game theoretical considerations of these examples.

which we denote by $St(\overrightarrow{a}_i)$.[14] Next, the payoff to player i from a *mixed strategy combination* $\overrightarrow{a} = (\overrightarrow{a}_1, ..., \overrightarrow{a}_n)$ is given as the expected payoff with respect to the probability distribution over the pure strategy combinations $(s_1, ..., s_n)$ induced by $\overrightarrow{a}$:

$$\sum_{t_1} \cdots \sum_{t_n} a_{1t_1} \cdot \ldots \cdot a_{nt_n} \cdot [\mathbf{g}_i(s_{t_1}, ..., s_{t_n})], \tag{5.10}$$

which we denote by $g_i(\overrightarrow{a})$.[15] Note that this $g_i(\overrightarrow{a})$ is a term. In the following, we denote $(\overrightarrow{a}_1, ..., \overrightarrow{a}_{i-1}, \overrightarrow{a}_{i+1}, ..., \overrightarrow{a}_n)$ by $\overrightarrow{a}_{-i}$, and $(\overrightarrow{a}_i; \overrightarrow{a}_{-i})$ means $\overrightarrow{a}$ itself.

Now we have the basic description of a game $\mathbf{g}$ with mixed strategies. Finally, we formulate the Nash equilibrium concept introduced by Nash [20] as a generalization of the maximin strategy of von Neumann [23], [24], which has been playing the central role in the literature of game theory. A *Nash equilibrium* is defined to be a mixed strategy combination $\overrightarrow{a} = (\overrightarrow{a}_1, ..., \overrightarrow{a}_n)$ satisfying the following formula:

$$\bigwedge \left\{ St(\overrightarrow{a}_i) \wedge \forall \overrightarrow{x}_i \left(St(\overrightarrow{x}_i) \supset g_i(\overrightarrow{a}) \geq g_i(\overrightarrow{x}_i; \overrightarrow{a}_{-i}) \right) : i = 1, ..., n \right\}, \tag{5.11}$$

where $\forall \overrightarrow{x}_i A(\overrightarrow{x}_i)$ means $\forall x_{i1} ... \forall x_{i\ell} A(x_{i1}, ..., x_{i\ell})$ and, later, $\exists \overrightarrow{x}_i A(\overrightarrow{x}_i)$ is used to denote $\exists x_{i1} ... \exists x_{i\ell} A(x_{i1}, ..., x_{i\ell})$. We denote the formula of (5.11) by $Nash_{\mathbf{g}}(\overrightarrow{a})$ or $Nash_{\mathbf{g}}(\overrightarrow{a}_1, ..., \overrightarrow{a}_n)$. Note that this is a formula relative to a specific game $\mathbf{g}$.

The prisoner's dilemma has a unique Nash equilibrium (C, C) even in mixed strategies (the formal counterpart is $((\mathbf{0}, \mathbf{1}), (\mathbf{0}, \mathbf{1}))$). The two-person game of Table 2, called "the Battle of Sexes", has three equilibria, (B, B), (M, M) and $((2/3, 1/3), (1/3, 2/3))$ (the formal counterparts are $((\mathbf{1}, \mathbf{0}), (\mathbf{1}, \mathbf{0}))$, $((\mathbf{0}, \mathbf{1}), (\mathbf{0}, \mathbf{1}))$ and $(([2/3], [1/3]), ([1/3], [2/3]))$.

5.2 Infinite regress of the knowledge of Final Decision Axioms and its solution

In a game $\mathbf{g}$, each player deliberates his and the others' strategy choices and may reach a final decision. The expression $D_i(\overrightarrow{a_i})$ describes a strategy decision $\overrightarrow{a_i}$ finally reached by a player. Recall that $D_i(\cdot)$ is an ℓ–ary predicate

[14] The orders of summation and product of over more than two variables should be specified in some manners. Under the axioms Φ_{of}, these are irrelevant, but we do not use these axioms in this section.

[15] This formula is based on von Neumann-Morgenstern's [25] expected utility theory (cf., Herstein-Milnor [7] for a simple axiomatization).

symbol, instead of a formula. We would like to characterize this "final decision" $D_i(\overrightarrow{a_i})$ operationally by the following four axioms: for $i, j = 1, ..., n$ (i, j may be the same),

$$D1: \ \forall \overrightarrow{x}_i \left(D_i(\overrightarrow{x}_i) \supset St(\overrightarrow{x}_i) \right);$$

$$D2: \ \forall \overrightarrow{x}_1 ... \forall \overrightarrow{x}_n \Big(\wedge_{j=1}^n D_j(\overrightarrow{x}_j) \supset \forall \overrightarrow{y}_i \big(St(\overrightarrow{y}_i) \supset g_i(\overrightarrow{x}) \geq g_i(\overrightarrow{y}_i; \overrightarrow{x}_{-i}) \big) \Big);$$

$$D3: \ \exists \overrightarrow{x}_i D_i(\overrightarrow{x}_i) \supset \exists \overrightarrow{x}_j D_j(\overrightarrow{x}_j);$$

$$D4: \ \forall \overrightarrow{x}_i \left(D_i(\overrightarrow{x}_i) \supset K_j(D_i(\overrightarrow{x}_i)) \right).$$

These mean: if $\overrightarrow{x}_i$ is a final decision for player i, then $D1$: it is a strategy; $D2$: given the others' final decisions $\overrightarrow{x}_{-i}$, $\overrightarrow{x}_i$ maximizes his payoff; $D3$: any other player j reaches also a final decision; and $D4$: all players know that player i reaches his final decision $\overrightarrow{x}_i$. Although each axiom has several formulae, we mean the conjunction of them by each. We denote $D1 \wedge D2 \wedge D3 \wedge D4$ by $D(1\text{–}4)$.

Axioms $D1$ and $D2$ is apparently related to Nash equilibrium, indeed,

$$D1, D2 \ \vdash_\omega \bigwedge_{i=1}^n D_i(\overrightarrow{a}_i) \supset Nash_g(\overrightarrow{a}). \tag{5.12}$$

Axiom $D3$ implies that he can make a final decision if and only if the others can make final decisions. In our axiomatization, each player makes his decision by considering his and the others' decisions. Axiom $D4$ is an epistemic condition and has not been explicitly discussed in the game theory literature. In fact, the explicit consideration of $D4$ leads to an infinite regress of the knowledge of these axioms.

Although those axioms are intended to determine $D_i(\overrightarrow{a}_i)$, we find, by looking at Axiom $D4$ carefully, that the above axioms are insufficient in the following sense. Axiom $D4$ requires that each player know his and the other player's final decisions, but this requirement could not be fulfilled unless the meaning of "final decisions" is given to the players. In fact, the meaning should be given by the above four axioms. Therefore we assume that each player knows these axioms, i.e., $K_i(D(1\text{–}4))$ for $i = 1, ..., n$. Then it holds that

$$D(1\text{–}4), \bigwedge_{\ell=1}^n K_\ell(D(1\text{–}4)) \vdash_\omega D_i(\overrightarrow{a}_i) \supset K_j K_t(D_i(\overrightarrow{a}_i)).$$

Again, we have a problem: player t in the mind of player j knows that $\overrightarrow{a}_i$ is a final decision for player i, but he is not given the meaning of "final decisions". Thus we need to assume $K_j K_t(D(1\text{–}4))$, but meet the same problem as above, that is, it holds in general that for any $K \in \mathsf{K}(m)$ and $m < \omega$,

$$\{L(D(1\text{–}4)) : L \in \bigcup_{t<m} \mathsf{K}(t)\} \vdash_\omega D_i(\overrightarrow{a}_i) \supset K(D_i(\overrightarrow{a}_i)). \tag{5.13}$$

Thus when we assume $L(D(1\text{–}4))$ for all L of depth up to $m-1$, it is required that the meaning of $D_i(\overrightarrow{a}_i)$ is known to the players in the sense of K of depth m. Hence we need to add $L(D(1\text{–}4))$ for L of depth m: we have the same problem as before. To avoid this problem, we assume $\{K(D(1\text{–}4)) : K \in \bigcup_{m<\omega} \mathsf{K}(m)\}$. Thus we meet an infinite regress, which forms the common knowledge of $D(1\text{–}4)$, i.e., $C(D(1\text{-}4))$.[16] We will solve this infinite regress.

Now we have the following proposition.

PROPOSITION 5.1.

1): $C(D1 \wedge D2 \wedge D4) \vdash_\omega \wedge_{i=1}^n D_i(\overrightarrow{a}_i) \supset C(Nash_{\mathbf{g}}(\overrightarrow{a}))$;

2): $C(D(1\text{–}4)) \vdash_\omega D_i(\overrightarrow{a}_i) \supset \exists \overrightarrow{x}_{-i} C(Nash_{\mathbf{g}}(\overrightarrow{a}_i; \overrightarrow{x}_{-i}))$.

PROOF. 1): By (5.12), $C(D1 \wedge D2 \wedge D4) \vdash_\omega \wedge_i D_i(\overrightarrow{a}_i) \supset Nash_{\mathbf{g}}(\overrightarrow{a})$. Let K be any element of $\bigcup_{m<\omega} \mathsf{K}(m)$. By Proposition 3.3.1), $C(D1 \wedge D2 \wedge D4) \vdash_\omega K(\wedge_i D_i(\overrightarrow{a}_i) \supset Nash_{\mathbf{g}}(\overrightarrow{a}))$. Using Proposition 3.1, we have

$$C(D1 \wedge D2 \wedge D4) \vdash_\omega \wedge_{i=1}^n K(D_i(\overrightarrow{a}_i)) \supset K(Nash_{\mathbf{g}}(\overrightarrow{a})). \tag{5.14}$$

Since $C(D1 \wedge D2 \wedge D4) \vdash_\omega \wedge_i D_i(\overrightarrow{a}_i) \supset \wedge_i K(D_i(\overrightarrow{a}_i))$ by (5.13), we have $C(D1 \wedge D2 \wedge D4) \vdash_\omega \wedge_i D_i(\overrightarrow{a}_i) \supset K(Nash_{\mathbf{g}}(\overrightarrow{a}))$. Since this holds for all $K \in \bigcup_{m<\omega} \mathsf{K}(m)$, we have $C(D1 \wedge D2 \wedge D4) \vdash_\omega \wedge_i D_i(\overrightarrow{a}_i) \supset C(Nash_{\mathbf{g}}(\overrightarrow{a}))$.
2): It follows from (1) that $C(D(1\text{–}4)) \vdash_\omega \wedge_{j\neq i} D_j(\overrightarrow{a}_j) \supset [D_i(\overrightarrow{a}_i) \supset C(Nash_{\mathbf{g}}(\overrightarrow{a}_i; \overrightarrow{a}_{-i}))]$. Using $L7$ and ($\exists$-*Rule*), we have $C(D(1\text{–}4)) \vdash_\omega \exists \overrightarrow{x}_{-i}(\wedge_{j\neq i} D_j(\overrightarrow{x}_j)) \supset [D_i(\overrightarrow{a}_i) \supset \exists \overrightarrow{x}_{-i} C(Nash_{\mathbf{g}}(\overrightarrow{a}_i; \overrightarrow{x}_{-i}))]$. Since $D3 \vdash_\omega D_i(\overrightarrow{a}_i) \supset \exists \overrightarrow{x}_{-i}(\wedge_{j\neq i} D_j(\overrightarrow{x}_j))$, we have $C(D(1\text{–}4)) \vdash_\omega D_i(\overrightarrow{a}_i) \supset [D_i(\overrightarrow{a}_i) \supset$

[16]We explained the necessity of each step from depth m to $m+1$ in a heuristic manner. In the finitary fragment of GL_ω, we can prove that the step of depth m cannot be derived from the previous one, using the depth lemma in Kaneko-Nagashima [12]. This lemma is not yet extended into the infinitary GL_ω.

$C(Nash_{\mathbf{g}}(\overrightarrow{a};\overrightarrow{a}_{-i}))]$, i.e., $C(D(1\text{–}4)) \vdash_\omega D_i(\overrightarrow{a}_i) \supset \exists \overrightarrow{x}_{-i} C(Nash_{\mathbf{g}}(\overrightarrow{a}_i; \overrightarrow{x}_{-i}))$. ■

The second assertion of Proposition 5.1 states that $D_i(a)$ implies $\exists \overrightarrow{x}_{-i} C(Nash_{\mathbf{g}}(\overrightarrow{a}_i; \overrightarrow{x}_{-i}))$. In fact, this formula can be regarded as the solution of $C(D(1\text{–}4))$ for some class of games. In the game of Table 2, either (pure) strategy is regarded as a candidate, since (B,B) and (M,M) are Nash equilibria. However, the independent choice of B for player 1 and M for player 2 leads to a nonNash point (B,M). To avoid this double cross, we restrict our attention to solvable games. A game $\mathbf{g}$ is called a *solvable* (in the sense of Nash [20]) iff the following holds:

$$\forall \overrightarrow{x}_1 \ldots \forall \overrightarrow{x}_n \left(\bigwedge_i (\exists \overrightarrow{y}_{-i} Nash_{\mathbf{g}}(\overrightarrow{x}_i; \overrightarrow{y}_{-i})) \supset Nash_{\mathbf{g}}(\overrightarrow{x}) \right). \tag{5.15}$$

This is satisfied by the game of Table 1 but not by that of Table 2. We denote this formula by *SOLV*. Of course, when the game $\mathbf{g}$ has a unique Nash equilibrium, this is satisfied.

By the expression $C(D(1\text{–}4))[A_1, ..., A_n]$, we mean the formula obtained from $C(D(1\text{–}4))$ by substituting each $A_i(\cdot)$ for every occurrence of $D_i(\cdot)$ in $C(D(1\text{–}4))$. If $\Gamma \vdash_\omega C(D(1\text{–}4))[A_1, ..., A_n]$, then $A_1, ..., A_n$ satisfy $C(D(1\text{–}4))$ under the assumptions Γ. The following lemma states that under the common knowledge of *SOLV*, the formulae of Proposition 5.1.2) satisfy the axioms $C(D(1\text{–}4))$.

LEMMA 5.2. *Let* $Sol_i(\overrightarrow{a}_i)$ *be* $\exists \overrightarrow{y}_{-i} C(Nash_{\mathbf{g}}(\overrightarrow{a}_i; \overrightarrow{y}_{-i}))$ *for* $i=1,...,n$. *Then* $C(SOLV) \vdash_\omega C(D(1\text{-}4))[Sol_1, ..., Sol_n]$.

PROOF. We prove only $C(SOLV) \vdash_\omega C(D2)[\, Sol_1, ..., Sol_n]$. First, since $\vdash_\omega \wedge_j\ Sol_j(\overrightarrow{a}_j) \supset \wedge_j C(\exists \overrightarrow{y}_{-j} Nash_{\mathbf{g}}(\overrightarrow{a}_j; \overrightarrow{y}_{-j}))$ by $(\exists_C)$ of Section 3, we have $\vdash_\omega \wedge_j Sol_j(\overrightarrow{a}_j) \supset \wedge_j \exists \overrightarrow{y}_{-j} Nash_{\mathbf{g}}(\overrightarrow{a}_j; \overrightarrow{y}_{-j})$. Thus we have

$$SOLV \vdash_\omega \wedge_j\ Sol_j(\overrightarrow{a}_j) \supset Nash_{\mathbf{g}}(\overrightarrow{a}_1, ..., \overrightarrow{a}_n).$$

Hence

$$SOLV \vdash_\omega \wedge_j Sol_j(\overrightarrow{a}_j) \supset \forall \overrightarrow{y}_i (St(\overrightarrow{y}_i) \supset g_i(\overrightarrow{a}) \geq g_i(\overrightarrow{y}_i; \overrightarrow{a}_{-i})).$$

By Proposition 3.3.2), we have $C(SOLV) \vdash_\omega C(D2)[Sol_1, ..., Sol_n]$.

For the other axioms, we do not need $C(SOLV)$, that is, $\vdash_\omega C(D1 \wedge D3 \wedge D4)$. ■

In Sections 5 and 6, the Barcan axiom, ($\wedge$-B_i), is used only in the proof that $Sol_i(\overrightarrow{a}_i)$ satisfies $D4$ or $C(D4)$. Without ($\wedge$-B_i), it cannot be proved that $Sol_i(\overrightarrow{a}_i)$ satisfies $D4$. This will be discussed in a separate paper.

The concept intended by $C(D(1\text{–}4))$ is the weakest one among those satisfying $C(D(1\text{–}4))$, since, otherwise, it would contain some properties additional to that given by $C(D(1\text{–}4))$. To require this idea, we impose the following axiom schema:

$$C(D(1\text{–}4)[A_1, ..., A_n]) \supset \forall \overrightarrow{x}_i(A_i(\overrightarrow{x}_i) \supset D_i(\overrightarrow{x}_i)),$$

where $A_1, ..., A_n$ are any formulae. We denote this by *WFD*. Since we proved in Lemma 5.2 that the premise of this formula is provable with $Sol_1, ..., Sol_n$ under the assumption of $C(SOLV)$, we have the $C(SOLV), WFD \vdash_\omega Sol_i(\overrightarrow{a}_i) \supset D_i(\overrightarrow{a}_i)$. This together with Proposition 5.1.2) implies the following theorem.

THEOREM 5.3. *$C(D(1\text{–}4)), C(SOLV), WFD \vdash_\omega D_i(\overrightarrow{a}_i) \equiv \exists y_{-i} C(Nash_{\mathrm{g}}(\overrightarrow{a}_i; \overrightarrow{y}_{-i}))$ for $i = 1, ..., n$.*

This theorem states that the final decision $\overrightarrow{a}_i$ is determined to be a Nash strategy with the common knowledge property. It is important to notice that the existential quantifier is outside the common knowledge operator. If it was $C(\exists y_{-i} Nash_{\mathrm{g}}(\overrightarrow{a}_i; \overrightarrow{y}_{-i}))$, which is implied by $\exists y_{-i} C(Nash_{\mathrm{g}}(\overrightarrow{a}_i; \overrightarrow{y}_{-i}))$ by ($\exists_C$), the existence of the other players' Nash strategies are simply required to be known. The formula $\exists y_{-i} C(Nash_{\mathrm{g}}(\overrightarrow{a}_i; \overrightarrow{y}_{-i}))$ requires player i to know specific Nash strategies for the other players. This difference is important for the subject of Section 6.

6. Applications to Game Theory II: Undecidability theorems on the playability of a game

The existence of a final decision, $\exists \overrightarrow{x}_i D_i(\overrightarrow{x}_i)$, is needed for each player to be able to make a final decision. By Theorem 5.3, this existence is equivalent to the existence of a Nash strategy with the common knowledge property, i.e., $\exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$. In classical game theory, the existence of a Nash equilibrium is proved by using Brouwer's fixed point theorem (cf., von Neumann [24] and Nash [20]). When the real number axioms are common knowledge, this existence proof implies $C(\exists \overrightarrow{x} Nash_{\mathrm{g}}(\overrightarrow{x}))$, where the existential quantifiers are in the scope of the common knowledge operator. There is a gap between the above two existential statements. In this section, we adopt the real closed field axioms as a particular choice of real number axioms, and show that although $C(\exists \overrightarrow{x} Nash_g(\overrightarrow{x}))$ is provable from the common knowledge of the real closed field axioms, $\exists \overrightarrow{x} C(Nash_g(\overrightarrow{x}))$ is formally undecidable, i.e., neither this existence statement nor its negation, $\neg\exists \overrightarrow{x} C(Nash_g(\overrightarrow{x}))$, is provable from the common knowledge of the real closed field axioms.

6.1 Real Closed Field Axioms and the existence of a Nash Equilibrium

The real closed field theory is defined by adding the following axioms to the ordered field axioms Φ_{of} :

$$\begin{gathered} \forall x \exists y(x \geq \mathbf{0} \supset (y^2 = x)); \\ \text{and} \\ \text{for any odd natural number } m, \\ \forall y_{m-1} ... \forall y_0 \exists x(x^m + y_{m-1}x^{m-1} + ... + y_1 x + y_0 = \mathbf{0}). \end{gathered} \tag{6.16}$$

We denote the union of Φ_{of} and the set of the formulae of (6.16) by Φ_{rcf}. The pair $(\mathcal{P}_{\mathrm{of}}, \Phi_{\mathrm{rcf}})$ is called the *real closed field theory,* where $\mathcal{P}_{\mathrm{of}}$ is the finitary nonepistemic fragment of $\mathcal{P}_\omega$ without including $D_1, ..., D_n$. Here we refer to Tarski's completeness theorem on the real closed field theory (cf., Rabin [21]): for any closed formula A in $\mathcal{P}_{\mathrm{of}}$, either $\Phi_{\mathrm{rcf}} \vdash_0 A$ or $\Phi_{\mathrm{rcf}} \vdash_0 \neg A$. Now we state two consequences of the completeness of $(\mathcal{P}_{\mathrm{of}}, \Phi_{\mathrm{rcf}})$.

The first one is: since *SOLV* is a formula in $\mathcal{P}_{\mathrm{of}}$, $\Phi_{\mathrm{rcf}} \vdash_0 SOLV$ or $\Phi_{\mathrm{rcf}} \vdash_0 \neg SOLV$ by Tarski's completeness theorem. Hence the solvability of a game g is decidable. This implies $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(SOLV)$ or $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(\neg SOLV)$. In Theorem 5.3, we can eliminate the assumption $C(SOLV)$ when we assume $C(\Phi_{\mathrm{rcf}})$ and g is chosen so that $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(SOLV)$.

The second one is more important. The standard existence proof of a Nash equilibrium for any finite game g with mixed strategies relies upon Brouwer's fixed point theorem (von Neumann [24] and Nash [20]). This implies that in the standard (real number) model of $(\mathcal{P}_{\mathrm{of}}, \Phi_{\mathrm{rcf}})$, the existence of a Nash equilibrium, $\exists \overrightarrow{x} Nash_g(\overrightarrow{x})$, is true. Since $(\mathcal{P}_{\mathrm{of}}, \Phi_{\mathrm{rcf}})$ is complete, we have $\Phi_{\mathrm{rcf}} \vdash_0 \exists \overrightarrow{x} Nash_g(\overrightarrow{x})$, which together with Proposition 3.5.1) implies the following.

PROPOSITION 6.1. *Let* g *be any n-person finite game. Then* $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(\exists \overrightarrow{x} Nash_g(\overrightarrow{x}))$.

Thus, in logic GL_ω, the existence of a Nash equilibrium is common knowledge if the real closed field axioms are common knowledge. Nevertheless, this is different from $C(\Phi_{\text{rcf}}) \vdash_\omega \exists \overrightarrow{x} C(Nash_{\mathbf{g}}(\overrightarrow{x}))$, which is required for a player in order to play the game $\mathbf{g}$ by Theorem 5.3. We would like to evaluate the provability of this assertion.

The following is the key result for such an evaluation, which is called the *term existence theorem*: for a set Γ of nonepistemic closed formulae and a nonepistemic formula A with no free variables in $\exists x_1 ... \exists x_\ell C(A(x_1, ..., x_m))$,

$$\begin{array}{c} C(\Gamma) \vdash_\omega \exists x_1 ... \exists x_m C(A(x_1, ..., x_m)) \\ \text{if and only if} \\ C(\Gamma) \vdash_\omega C(A(t_1, ..., t_m)) \text{ for some closed terms } t_1, ..., t_m. \end{array} \tag{6.17}$$

This term existence theorem will be proved in Part II, using the cut-elimination theorem for GL_ω. This theorem tells us that we should distinguish between the mere knowledge of the existence and the specific objects having the common knowledge of property A.

As the direct application of (6.17) to our game theoretical problem, we have

$$\begin{array}{c} C(\Phi_{\text{rcf}}) \vdash_\omega \exists \overrightarrow{x} C(Nash_{\mathbf{g}}(\overrightarrow{x})) \\ \text{if and only if} \\ C(\Phi_{\text{rcf}}) \vdash_\omega C(Nash_{\mathbf{g}}(\overrightarrow{t})) \text{ for some closed term vector } \overrightarrow{t}. \end{array} \tag{6.18}$$

Thus, for the specific existence, we need probability vectors $\overrightarrow{t}_i = (t_{i1}, ..., t_{i\ell})$, each component of which is represented as a closed term. In the present language together with the ordered field axioms Φ_{of}, for any closed term t there is a rational number r such that $\Phi_{\text{of}} \vdash t = [r]$. Informally speaking, (6.18) implies that there should exist a Nash equilibrium in rational numbers. However, this does not always hold for games with more than two players.

6.2 Undecidability theorems on the playability of a game

Consider the following three-person game given by both Tables 3 and 4:

	β_1	β_2
α_1	(0,0,1)	(1,0,0)
α_2	(1,1,0)	(2,0,8)

γ_1

Table 3

	β_1	β_2
α_1	(2,0,9)	(0,1,1)
α_2	(0,1,1)	(1,0,0)

γ_2

Table 4

In this game, each player has two pure strategies, and the tables mean that when the players choose pure strategies, say, $\alpha_1, \beta_2, \gamma_2$, the right upper vector

(0,1,1) of Table 4 gives payoffs to the players. This game has no Nash equilibrium in pure strategies, but has a unique Nash equilibrium $((p, 1-p), (q, 1-q), (r, 1-r))$ in mixed strategies, where

$$p = (30 - 2\sqrt{51})/29,\ q = (2\sqrt{51} - 6)/21 \text{ and } r = (9 - \sqrt{51})/12.$$

The probability weights in equilibrium are irrational numbers. Therefore those probabilities are not represented as closed terms in our language. Therefore it follows from (6.18) that $C(\Phi_{\mathrm{rcf}}) \vdash_\omega \exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$ is not the case.

In fact, the negation of this existential assertion $C(\Phi_{\mathrm{rcf}}) \vdash_\omega \neg\exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$ is equivalent to $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(\neg\exists \overrightarrow{x} Nash_{\mathrm{g}}(\overrightarrow{x}))$, as was stated in (4.8). Hence Proposition 6.1 implies that it is not the case that $C(\Phi_{\mathrm{rcf}}) \vdash_\omega \neg\exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$.

In sum, we have the following theorem.

THEOREM 6.2. *[Formal Undecidability I] Let* g *be the three-person game given by Tables* 3 *and* 4. *Then neither* $C(\Phi_{\mathrm{rcf}}) \vdash_\omega \exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$ *nor* $C(\Phi_{\mathrm{rcf}}) \vdash_\omega \neg\exists \overrightarrow{x} C(Nash_{\mathrm{g}}(\overrightarrow{x}))$.

As was stated in (6.18), the condition for a player to find a Nash strategy is that there is a Nash equilibrium in closed terms. He can verify whether each closed term vector satisfies the Nash condition. Therefore if there is a Nash equilibrium in closed terms, he would eventually find a Nash equilibrium. However, when there is no Nash equilibrium in closed terms such as in the game of Tables 3 and 4, he continues the verification of whether each candidate satisfies the Nash condition. Each player does not have the knowledge of the space of closed terms, more generally, he does not have knowledge about the language as a whole he is using. Therefore he should continue to search a Nash equilibrium, and cannot know whether there is a Nash equilibrium or not.

For the above three-person game, our undecidability result would become a decidability one if we introduce a function symbol and some axiom to allow the radical expression $\sqrt{\ }$. Thus the above undecidability result depends upon the choice of a language. The point of the theorem is, however, that the players cannot notice the necessity of an extension of the language, since neither the positive nor negative statement is known to them.

The property that a Nash equilibrium involves irrational numbers is general for games with more than two players, except some degenerate cases. In fact, it is proved in Bubelis [5] that any algebraic real number in $[0,1]$ occurs in a Nash equilibrium for some three-person game with finite numbers of pure strategies.[17] Thus the problem of obtaining the decidability result in the general case is not so simple as in the case mentioned in the above paragraph for the particular game. This will be discussed in Kaneko [10].

In Section 5, our concern was the determination of final decision predicate $D_i(\overrightarrow{a}_i)$. Under axioms $C(D(1\text{–}4)), C(SOLV)$ and WFD, final decision $D_i(\overrightarrow{a}_i)$ coincides with $\exists \overrightarrow{x}_{-i} C(Nash_{\mathrm{g}}(\overrightarrow{a}_i; \overrightarrow{x}_{-i}))$. Noting that when $C(\Phi_{\mathrm{rcf}})$ is assumed, $C(SOLV)$ is not necessary, the playability of a game g is directly stated as

$$\text{whether or not } C(D(1\text{–}4)), WFD, C(\Phi_{\mathrm{rcf}}) \vdash_\omega \exists \overrightarrow{x}_i D_i(\overrightarrow{x}_i). \tag{6.19}$$

In fact, we obtain a formal undecidability on $\exists \overrightarrow{x}_i D_i(\overrightarrow{x}_i)$.

THEOREM 6.3. *[Formal Undecidability II] Let* g *be the three-person game given by Tables* 3 *and* 4. *Then*

$$\text{neither } C(D(1\text{–}4)), WFD, C(\Phi_{\mathrm{rcf}}) \vdash_\omega \exists \overrightarrow{x}_i D_i(\overrightarrow{x}_i)$$

$$\text{nor } C(D(1\text{–}4)), WFD, C(\Phi_{\mathrm{rcf}}) \vdash_\omega \neg\exists \overrightarrow{x}_i D_i(\overrightarrow{x}_i).$$

First, we prove the following lemma.

LEMMA 6.4. *Let $\mathcal{P}_\omega^\#$ be the set of formulae in $\mathcal{P}_\omega$ without including D_1, ..., D_n. Let* g *be a solvable game. Then the theory $(\mathcal{P}_\omega, C(D(1\text{–}4)), WFD, C(\Phi_{\mathrm{rcf}}))$ is a conservative extension of the theory $(\mathcal{P}_\omega^\#, C(\Phi_{\mathrm{rcf}}))$.*

PROOF. We denote, by $D^\#(1\text{–}4)$ and $WFD^\#$, the formulae which are obtained from $D(1\text{–}4)$ and WFD by substituting $Sol_i(\cdot)$ for $D_i(\cdot)$ for $i = 1, ..., n$. Then we can prove that $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(D^\#(1\text{–}4))$ and $C(\Phi_{\mathrm{rcf}}) \vdash_\omega WFD^\#$ in $\mathcal{P}_\omega^\#$. Indeed, consider $C(\Phi_{\mathrm{rcf}}) \vdash_\omega WFD^\#$ in $\mathcal{P}_\omega^\#$: Proposition 5.1 together with the substitution of $Sol_i(\cdot)$ for $D_i(\cdot)$ implies $C(\Phi_{\mathrm{rcf}}) \vdash_\omega C(D(1\text{–}4)[A_1, ..., A_n]) \supset (A_i(\overrightarrow{a}_i) \supset Sol_i(\overrightarrow{a}_i))$, that is, $C(\Phi_{\mathrm{rcf}}) \vdash_\omega WFD^\#$ in $\mathcal{P}_\omega^\#$.

Suppose $C(D(1\text{–}4)), WFD, C(\Phi_{\mathrm{rcf}}) \vdash_\omega A$, where A is a formula in $\mathcal{P}_\omega^\#$. Then there is a proof P of A from $C(D(1\text{–}4)), WFD, C(\Phi_{\mathrm{rcf}})$ in GL_ω. We substitute Sol_i for all occurrences of D_i in P, and get a proof $P^\#$ of A from

[17] Lemke-Howson [15] gave a finite algorithm to find a Nash equilibrium for a two-person game with mixed strategies, which implies the existence of a Nash equilibrium in rational numbers. Therefore undecidability fails since the existential formula is provable for any two-person game. However, if we formulate the real closed field theory based on only + and ·, then we obtain an undecidability result even in the two-person case.

$C(D^{\#}(1\text{–}4)), WFD^{\#}, C(\Phi_{\rm rcf})$, that is, $C(D^{\#}(1\text{–}4)), WFD^{\#}, C(\Phi_{\rm rcf}) \vdash_\omega A$. However, since the first two premises are derived from $C(\Phi_{\rm rcf})$ in $\mathcal{P}^{\#}_\omega$, this proof can be regarded as a proof of A from $C(\Phi_{\rm rcf})$ in $\mathcal{P}^{\#}_\omega$. ■

PROOF OF THEOREM 6.3. On the contrary, suppose $C(D(1\text{–}4)), WFD, C(\Phi_{\rm rcf}) \vdash_\omega \exists \vec{x}_i D_i(\vec{x}_i)$. By Theorem 5.3, we have $C(D(1\text{–}4)), WFD, C(\Phi_{\rm rcf}) \vdash_\omega \exists \vec{x} C(Nash_{\rm g}(\vec{x}))$. By Lemma 6.4, we have $C(\Phi_{\rm rcf}) \vdash_\omega \exists \vec{x} C(Nash_{\rm g}(\vec{x}))$, which is impossible by Theorem 6.2.

If $C(D(1\text{–}4)), WFD, C(\Phi_{\rm rcf}) \vdash_\omega \neg\exists \vec{x}_i D_i(\vec{x}_i)$, we have $C(D(1\text{–}4)), WFD, C(\Phi_{\rm rcf}) \vdash_\omega \neg\exists \vec{x} C(Nash_{\rm g}(\vec{x}))$ in the same way as above. By Lemma 6.4, we have $C(\Phi_{\rm rcf}) \vdash_\omega \neg\exists \vec{x} C(Nash_{\rm g}(\vec{x}))$, which is impossible by Theorem 6.2. ■

7. Conclusions

This paper provided the logic framework for the investigations of game theoretical problems, and showed two applications. The first application is an epistemic axiomatization of Nash equilibrium, and the second is the undecidability on the playability of a game. The first is still a game theoretical problem, though it was discussed in the game logic framework. The second is also a game theoretical problem, but it can be regarded as a logic problem as well in that it is a meta-result. It is important that the latter was raised by the former. Therefore, these form a result belonging to both game theory and mathematical logic.

To obtain the undecidability results, we used the term existence theorem, which is a metatheorem on provability. It is difficult to prove such metatheorems in the present Hilbert style formulation. In Part II of this paper, we reformulate the game logic framework in the Gentzen style sequent calculus, and prove the cut-elimination theorem for it. By the cut-elimination theorem, we prove the term existence theorem and the converse of the Proposition 2.2 (faithful representation). The Gentzen style formulation and the cut-elimination theorem will provide other deeper results. These are the subjects of Part II.

From the viewpoints of logic as well as of game theory, the epistemic axiomatization of Nash equilibrium in Section 5 needs more discussions. Game theoretical discussions are found in Kaneko-Nagashima [11] and Kaneko [9]. Proof theoretical evaluations of the epistemic axiomatization will be discussed also in [9].

As was mentioned, the undecidability results of Section 6 depend upon the choice of constants or function symbols. If more constants are introduced to describe all real algebraic numbers, then we obtain the decidability results for any finite game. There still remain important problems in this direction from the viewpoint of both logic and game theory. These will be discussed in Kaneko [10].

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