Exploring the relationship between the hedging strategies based on coherent risk measures and the martingale probabilities via optimization approach

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Abstract

An application of the duality theory of linear optimization leads to the well-known arbitrage pricing theorems of financial mathematics, namely, the equivalence between the absence of arbitrage and the existence of an equivalent martingale probability measure. The prices of contingent claims can then be calculated based on the set of martingale probability measures. Especially, in the incomplete market which has more than one equivalent probability measures, an interval for the no-arbitrage price is obtained rather than a single value.

In this thesis, we address a problem of pricing contingent claims in a discrete model of the incomplete market by extending the hedging concept. A narrower no-arbitrage interval of the contingent claim price is obtained by replacing the traditional no-risk condition with a new idea associated with coherent risk measures. The price interval can be calculated by solving a pair of linear programs where the decision variables vary over a subset of martingale probability measures which is uniquely characterized by a given coherent risk measure.

Some computational results are also reported, showing that the no-arbitrage interval may turn into a single point if an adequate coherent risk measure is employed. Such a single value is considered as a fair price of the contingent claim since the seller and the buyer face to the same risk if the contingent claim is traded at that price.

Key words: arbitrage; martingales; acceptance set; coherent risk measures; hedging; pricing of the contingent claims.
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1 Introduction

Contingent claim is an asset or security whose value depends upon other assets or numerical indices. The pricing theory of financial contingent claims has a long and illustrious history which starts with the revolutionary developments in 1973 when F. Black and M. Scholes presented the first completely satisfactory equilibrium option pricing model, and then R. Merton extended their model in several important ways. Since their seminal works, numerous researches have committed to establish the mathematical foundation of the pricing of contingent claims.

Most researches model the uncertainty of the asset market in either of two ways: the continuous time model and the discrete time model. The former assumes that investors can trade a set of assets in the market at any moment they would like to do and there are infinitely many possible states, while the latter assumes that the trade can occur only at finite times in the time span and the possible states is usually set to be finite. More specifically, in the continuous time model the dynamics of the asset price is modeled via some continuous time stochastic process.

As Black and Scholes succeeded in driving the analytical formula of the price of European call option, the continuous model can provide an explicit formula by assuming that the underlying asset price follows a specific underlying stochastic process such as the geometric Brownian motion. One of the major drawbacks of the continuous model is, however, that the assumption can be too restrictive to adopt the model to the real market data. In addition, most of the continuous models can treat only a few underlying assets simultaneously.

In order to enable the model to fit the real situation, several directions have been considered for avoiding restrictive assumptions on the underlying distribution. One direction is to introduce a general class of stochastic processes such as the Levi process (e.g., [22]). See Carr et al. [4] for an example. Another direction is to abandon the parametric distribution. Among such researches, some suppose that partial information of the distribution
such as the moments is available (e.g., [16, 3, 8, 15]).

On the other hand, the discrete model can deal with a general distribution by way of the scenario tree approximation and can treat multiple underlying assets simultaneously in contrast with the continuous model. For example, Ritchken [18] formulates simple linear programming problems for computing a pair of upper and lower bounds on a European style contingent claim in a single period model, and Ritchken and Kuo [19] and Basso and Pianca [2] extend the approach so as to reduce the gap of the pricing bounds by introducing the risk attitude of investors. King ([13], [14]) points out, however, that such a reduction cannot explain the pricing unless the gap vanishes because the upper bound implies the lower bound of the seller’s price for selling the asset, i.e., ask price, while the lower bound implies the upper bound of the buyer’s price for buying the asset, i.e., bid price.

Except for Cox, Ross and Rubinstein [6] and several interest rate tree models such as Hull [12], the use of the generalized discrete models may not be much popular in practice than the continuous models. However, those models provide a clear insight of the pricing of the contingent claims (see, e.g., [17, 13]).

In addition, how to price contingent claims in incomplete market is still a big remaining problem for the financial theory because the market in the real world is obviously incomplete. One of the advantages of the discrete model is that it can treat the incomplete market situation in a similar manner to the complete one. King [13] argues the contingent claim pricing problem on a discrete model, and proves that the contingent claim must be traded at a price in a single interval which is calculated based on the martingale probability measures. The interval is referred to as the no arbitrage interval because any price out of this interval will induce some arbitrage opportunities for either buyer or seller.

In this thesis, we investigate the length of the no arbitrage interval for the price of contingent claims in incomplete market and how it is decided by hedging with the risk measure one uses. We show a narrower interval of contingent claim’s prices by replacing
the traditional no-risk condition on the arbitrage with the one which is based on a coherent
risk measure, which is first introduced by Artzner et al.[1]. More specifically, we show
that the price of contingent claims can be calculated based on a subset of martingale
probability measures which is decided by the coherent risk measure one employed.

The contribution of this thesis is twofold. Firstly, we extend the fundamental theory
of asset pricing to a much more general version by replacing the traditional no-risk con-
dition in the definition of the arbitrage with that associated with coherent risk measures.
Secondly, we propose an explanation for decreasing and vanishing the gap between the ask
and bid prices, which suggests that there is a way to price contingent claims in incomplete
market in a manner of complete market consistent with the coherent risk measures.

The structure of this thesis is as follows. In Chapter 2, the notations and assumptions
for our discussion are described, and the known results on the structure of the fundamental
theorem of asset pricing are summarized. Chapter 3 is devoted to introducing the concepts
of the acceptance set and coherent risk measures following [1], and a modification of the
definition of the arbitrage is proposed on the basis of the risk measures other than the
maximum loss which has been usually applied. One of our main results is also proved,
and a geometric example illustrating the result is provided. In the following two sections,
we apply our idea to the hedging and pricing problem of two popular types of contingent
claims: European style and American style. In Chapter 6, numerical examples are given,
presenting how to implement the model we proposed. Finally, Chapter 7 provides a
summary and some remarks, concluding the thesis.

2 Preliminaries

In this chapter, we discuss the discrete time version of the problem of hedging and pricing
contingent claims. Then, we introduce the fundamental theorem of asset pricing.
2.1 Notations and basic assumptions

We first introduce the basic framework for modeling the uncertainty of asset values. All random quantities here will be defined on a finite probability space \((\Omega, \mathcal{F}, P^*)\) whose atom is a sequence of real-valued vectors (asset values) over discrete time period \(t = 0, \ldots, T\).

Let \(\mathcal{N}_t\) be the set of all nodes at depth \(t\) in the scenario tree, \(t = 0, 1, \ldots, T\), and let \(\Omega = \bigcup_{0 \leq t \leq T} \mathcal{N}_t\). The initial node is denoted by \(n = 0\), which means \(\mathcal{N}_0 = \{0\}\). In the scenario tree, every node \(n \in \mathcal{N}_t\) with \(t \in \{1, \ldots, T\}\) has a unique parent node in \(\mathcal{N}_{t-1}\) denoted by \(a(n)\), and every node \(n \in \mathcal{N}_t\) with \(t \in \{0, \ldots, T-1\}\) has a nonempty set of child nodes in \(\mathcal{N}_{t+1}\), denoted by \(c(n)\). The original probability distribution \(P^*\) is modeled by attaching weights \(p^*_n > 0\) to each terminal node \(n \in \mathcal{N}_T\) so that \(\sum_{n \in \mathcal{N}_T} p^*_n = 1\). For each non-terminal node, the probability is attached as follows:

\[
p^*_n = \sum_{m \in c(n)} p^*_m, \quad \forall n \in \mathcal{N}_t, \quad t = 0, \ldots, T-1.
\]  

(2.1)

We suppose that the market has \(J + 1\) tradable securities indexed by \(j = 0, \ldots, J\) whose prices at node \(n\) are denoted by the vector \(S_n := (s^0_n, \ldots, s^J_n)^\top\). Also, we assume that one of the securities, say security 0, always has positive value, i.e., \(s^0_n > 0\) for all \(n \in \Omega\). Such a security is referred to as numeraire.

Let us denote the discount rate by \(\delta_n := (1/s^0_n)\) and let \(Z_n := (z^0_n, \ldots, z^J_n)^\top =: \delta_n(s^0_n, \ldots, s^J_n)^\top\) for every \(n \in \Omega\). At each node \(n\), \(Z_n\) is the relative price vector with respect to security 0. Obviously, \(z^0_n = 1\) for all \(n \in \Omega\). For \(t \in \{0, \ldots, T\}\), \(Z_t\) is a random variable on \(\mathcal{N}_t\) with \(Z_t(n) = Z_n\) for each \(n \in \mathcal{N}_t\).

\(\{\Theta_n = (\theta^0_n, \ldots, \theta^J_n)^\top : n \in \Omega\}\) is called a trading strategy where \(\theta^j_n\) \((j = 0, \ldots, J)\) denotes the amount of security \(j\) held by the investor at node \(n \in \Omega\). For \(t \in \{0, \ldots, T\}\), \(\Theta_t\) is a random variable on \(\mathcal{N}_t\) with \(\Theta_t(n) = \Theta_n\) for each \(n \in \mathcal{N}_t\).

**Definition 2.1** We say a trading strategy \(\{\Theta_n = (\theta^0_n, \ldots, \theta^J_n)^\top : n \in \Omega\}\), or \(\Theta\), for simplicity, is self-financing if it satisfies \(Z_n^\top \Theta_n = Z_n^\top \Theta_{a(n)}\) for each \(n \in \mathcal{N}_t, \ t \in \{1, \ldots, T\}\).
2.2 Fundamental theorem of asset pricing

Before summarizing the fundamental theorem of asset pricing, we first introduce the definition of traditional arbitrage strategy. In finance theory, an arbitrage means a “free lunch”—an investment that makes a profit without risk.

Definition 2.2 Traditional arbitrage strategy: Arbitrage is a self-financing trading strategy \( \Theta \) that begins with zero initial value at time 0, maintains a non-negative value at each terminal node \( n \in \mathcal{N}_T \) and has a positive expected value at the maturity date \( T \).

Mathematically, arbitrage is a trading strategy \( \Theta \) such that

\[
\begin{align*}
Z_0^\top \Theta_0 &= 0 \\
Z_n^\top [\Theta_n - \Theta_{a(n)}] &= 0 \quad (n \in \mathcal{N}_t, \ t \geq 1) \\
Z_n^\top \Theta_n &\geq 0 \quad (n \in \mathcal{N}_T) \\
\sum_{n \in \mathcal{N}_T} p_n^* Z_n^\top \Theta_n &> 0.
\end{align*}
\] (2.2)

The market is said to be arbitrage free if there is no chance of arbitrage in the market. In an arbitrage free market, if adding one security \( Y \) that is traded at price \( Y_0 \) keeps the market arbitrage free, we say \( Y_0 \) is a no arbitrage price of security \( Y \).

Harrison and Kreps [9] proved that the absence of arbitrage is essentially equivalent to the existence of an equivalent martingale probability measure \( Q \) such that

1) \( Q \) and \( P^* \) agree on impossible events, namely \( q_n = 0 \) if and only if \( p_n^* = 0 \) for all \( n \in \Omega \).

2) \( \{ Z_t : 0 \leq t \leq T \} \) is a martingale process under the probability measure \( Q \), that is

\[ Z_n = \sum_{m \in \mathcal{C}(n)} q_m Z_m \] for any \( n \in \mathcal{N}_t, \ t < T \).

Further, the no arbitrage price of security \( Y \) is given by the expected value of \( Y \) at the maturity date \( T \) based on an arbitrarily chosen equivalent martingale probability measure.

Theorem 2.3 (Fundamental theorem of asset pricing;)

1) There is no arbitrage if and only if there is a martingale probability measures equivalent to \( P^* \).
2) In the arbitrage free market, the lower and upper bounds for the price $Y_0$ of security $Y$ are given, respectively, by

$$\frac{1}{\delta_0} \min_{Q \in \mathcal{M}} \sum_{n \in N_T} q_n \delta_n Y_n \quad \text{and} \quad \frac{1}{\delta_0} \max_{Q \in \mathcal{M}} \sum_{n \in N_T} q_n \delta_n Y_n,$$

where $\mathcal{M}$ is the set of martingale probability measures equivalent to $P^*$, $\bar{\mathcal{M}}$ is the closure of $\mathcal{M}$, and $Y_n$ is the future value of contingent claim $Y$ corresponding to each node $n \in N_T$.

**Proof:** See King [13] for example.

If there is a unique martingale probability measure equivalent to the original probability measure $P^*$, the market is said to be complete. In this case, the no arbitrage price of security $Y$ is given by $\frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n$ with $Q \in \mathcal{M}$.

Otherwise, the market is said to be incomplete. If the price $Y_0$ of contingent claim $Y$ satisfies $Y_0 \leq \frac{1}{\delta_0} \min_{Q \in \mathcal{M}} \sum_{n \in N_T} q_n \delta_n Y_n$, there exists an arbitrage for the buyer, on the other hand, $Y_0 \geq \frac{1}{\delta_0} \max_{Q \in \bar{\mathcal{M}}} \sum_{n \in N_T} q_n \delta_n Y_n$ induces an arbitrage for the seller. Thus the no arbitrage interval of $Y$ is given by

$$\left(\frac{1}{\delta_0} \min_{Q \in \mathcal{M}} \sum_{n \in N_T} q_n \delta_n Y_n, \frac{1}{\delta_0} \max_{Q \in \bar{\mathcal{M}}} \sum_{n \in N_T} q_n \delta_n Y_n\right).$$

3 Acceptance set, risk measure and arbitrage

In this chapter, we introduce concepts of the acceptance set and coherent risk measure, following Artzner et al.[1]. Given the concept of the acceptance set, we propose a definition of general type of arbitrage. The condition for the absence of arbitrage which is defined based on the coherent risk measures is also given in this chapter.

3.1 General type arbitrages

Let $X$ be the value of a portfolio at the maturity date $T$, then $X$ is a random variable on the probability space $(\mathcal{N}_T, \mathcal{F}_T, P^*)$. 
We define the *acceptance set* of the investor by the set of net worths $X$ that are acceptable to the investor and denote it by $\mathcal{A}$. It is the range of outcomes the investor is willing to take. Given an acceptance set $\mathcal{A}$, a trading strategy $\Theta$ is said to be *acceptable* if $Z_T^\top \Theta_T \in \mathcal{A}$.

Based on the idea of acceptance set, we define the general type of arbitrage as follows:

**Definition 3.1 (General definition of arbitrage:)** A self-financing trading strategy $\Theta$ is an arbitrage if it begins with zero initial value at time 0, maintains an acceptable net worth at each terminal node $n \in \mathcal{N}_T$, and has a positive expected value at the maturity date $T$. Mathematically, the generalized version of arbitrage associated with an acceptance set $\mathcal{A}$ is a trading strategy $\Theta$ such that:

$$
\begin{align*}
Z_0^\top \Theta_0 &= 0 \\
Z_n^\top [\Theta_n - \Theta_{a(n)}] &= 0 \quad (n \in \mathcal{N}_t, t = 1, \ldots, T) \\
Z_n^\top \Theta_n &\in \mathcal{A} \quad (n \in \mathcal{N}_T) \\
\sum_{n \in \mathcal{N}_T} p_n^* Z_n^\top \Theta_n &> 0.
\end{align*}
$$

If $\mathcal{A} = \mathcal{L}_+$, with $\mathcal{L}_+ = \{ X \mid \text{for each } \omega \in \Omega, X(\omega) \geq 0 \}$, the arbitrage associated with $\mathcal{A}$ is coincides with the traditional arbitrage.

If the acceptance set $\mathcal{A}$ contains the set $\mathcal{L}_+$, then the no risk condition for the new type of arbitrage associated with $\mathcal{A}$ is weaker than the no risk condition for the traditional arbitrage.

**Definition 3.2 (Risk measure associated with an acceptance set (Artzner et al.[1]):)** Risk measure $\rho$ is a mapping from the set of all net worths $X$ to $\mathbb{R}$. The risk measure $\rho_\mathcal{A}$ associated with an acceptance set $\mathcal{A}$ is defined by

$$
\rho_\mathcal{A}(X) = \inf \{ c \mid c + X \in \mathcal{A} \}.
$$

**Definition 3.3 (Acceptance set associated with a risk measure:)** Given a risk
measure $\rho$, the acceptance set associated with $\rho$, denoted by $A_\rho$, is defined by

$$A_\rho = \{ X \mid \rho(X) \leq 0 \}.$$

### 3.2 Coherent risk measures

In this thesis, we focus on a class of coherent risk measures, which are first introduced by Artzner et al.[1].

**Definition 3.4 (Coherent risk measure:)** A risk measure $\rho$ is said to be coherent if $\rho$ has the following four properties.

a) **Translation invariance:** For all net worth $X$ and real number $c$, $\rho(X+c) = \rho(X) - c$.

b) **Subadditivity:** For all net worths $X_1$ and $X_2$, $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

c) **Positive homogeneity:** For all $\lambda \geq 0$ and net worth $X$, $\rho(\lambda X) = \lambda \rho(X)$.

d) **Monotonicity:** For all net worths $X$ and $Y$ with $X \leq Y$, it holds that $\rho(Y) \leq \rho(X)$.

Artzner et al.[1] proved that if $\rho$ is a coherent risk measure, then the acceptance set $A_\rho$ is closed and has the following four properties:

A) $A_\rho$ contains $L_+$.  

B) $A_\rho$ does not intersect the set $L_- = \{ X \mid \text{ for each } \omega \in \Omega, X(\omega) < 0 \}$. 

C) $A_\rho$ is convex. 

D) $A_\rho$ is a cone.

Further, if an acceptance set $A$ satisfies the properties A), B), C), D), then the risk measure $\rho_A$ associated with $A$ is a coherent risk measure.

Artzner et al.[1] also showed that any risk measure $\rho$ corresponds to a subset of probability measures $\mathcal{P}_\rho$ defined as $\mathcal{P}_\rho = \{ P \mid P$ is a probability measure on $\Omega$ and $E^P(-X) \leq \rho(X)$ for all net worth $X \},$ and $\rho(X) = \sup \{ E^P(-X) \mid P \in \mathcal{P}_\rho \}$ for all net worths $X$.

Further, Artzner et al.[1] proved that when the risk measure $\rho$ is coherent, the set $\mathcal{P}_\rho$ is convex and closed. Since $E^P(-X)$ is continuous in $P$ for any fixed $X$, one has $\rho(X) = \sup \{ E^P(-X) \mid P \in \mathcal{P}_\rho \} = \max \{ E^P(-X) \mid P \in \mathcal{P}_\rho \}$. 

8
3.3 Absence of arbitrage

It can be seen that when a coherent risk measure $\rho$ is employed, the acceptance set $\mathcal{A}_\rho = \{ X \mid \rho(X) = \max\{ \sum_{n \in \mathcal{N}_T} E^P(-X) \} \leq 0 \}$. Then the arbitrage corresponding to the acceptance set $\mathcal{A}_\rho$ can be defined as:

**Definition 3.5 ($\rho$-arbitrage):** For any coherent risk measure $\rho$, an arbitrage defined based on the acceptance set $\mathcal{A}_\rho$, which is referred to as $\rho$-arbitrage, is a trading strategy $\Theta$ that satisfies

1) $Z_0 = 0$.
2) $Z_n^\top \Theta_n = Z_n^\top \Theta_{\alpha(n)}$ for each $n \neq 0$.
3) $\rho(Z_T^\top \Theta_T) = \max\{ \sum_{n \in \mathcal{N}_T} p_n(-Z_n^\top \Theta_n) \mid P \in \mathcal{P}_\rho \} \leq 0$.
4) $E^P\left(Z_T^\top \Theta_T\right) = \sum_{n \in \mathcal{N}_T} p_n^* Z_n^\top \Theta_n > 0$.

The following theorem shows the conditions for the absence of arbitrage in the meaning of arbitrage associated with coherent risk measure $\rho$.

**Theorem 3.6** For any coherent risk measure $\rho$, let $\mathcal{Q} = \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$, where $\text{int}(\mathcal{P}_\rho)$ is the set of relative interior points of $\mathcal{P}_\rho$, and $\mathcal{M}$ is the set of all martingale probability measures on $\Omega$ that are equivalent to the original probability measure $P^\ast$. Under the assumption that $P^\ast \in \text{int}(\mathcal{P}_\rho)$, there is no $\rho$-arbitrage if and only if the set $\mathcal{Q}$ is nonempty.

To prove Theorem 3.6, we first prove that the result of Theorem 3.6 holds when $\mathcal{P}_\rho$ is a polytope.

**Lemma 3.7** For any coherent risk measure $\rho$, let $\mathcal{Q} = \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$, where $\text{int}(\mathcal{P}_\rho)$ is the set of relative interior points of $\mathcal{P}_\rho$, and $\mathcal{M}$ is the set of all martingale probability measures on $\Omega$ that are equivalent to the original probability measure $P^\ast$. Under the assumptions that $\mathcal{P}_\rho$ is a polytope and $P^\ast \in \text{int}(\mathcal{P}_\rho)$, there is no $\rho$-arbitrage if and only if the set $\mathcal{Q}$ is nonempty.
Proof of Lemma 3.7: Let us focus on the following linear programming.

\[
\max_{\Theta} \sum_{n \in \mathcal{N}_T} p_n^* Z_n^\top \Theta_n \\
\text{s.t.} \\
Z_0^\top \Theta_0 = 0 \\
Z_n^\top [\Theta_n - \Theta_{a(n)}] = 0 \\
\rho(Z_T^\top \Theta_T) = \max \{ \sum_{n \in \mathcal{N}_T} p_n (-Z_n^\top \Theta_n) \mid P \in \mathcal{P}_\rho \} \leq 0 \\
\tag{3.2}
\]

The optimal value of (3.2) is nonnegative, since \( \Theta = 0 \) is a feasible solution of (3.2) with objective value 0. By the definition of \( \rho \)-arbitrage, a positive optimal value for this linear program corresponds to a \( \rho \)-arbitrage. Thus there is no \( \rho \)-arbitrage if and only if the optimal value of (3.2) is zero.

Furthermore, \( k \cdot \Theta^* \) is also a feasible solution of (3.2) for any positive number \( k \) and any feasible solution \( \Theta^* \) of (3.2). Therefore, the linear program (3.2) has either the optimal value 0 or an unbounded objective function value. In other words, there is no \( \rho \)-arbitrage if and only if the linear program (3.2) is feasible and has a finite optimal value.

Since \( \mathcal{P}_\rho \) is a polytope, it has a finite number of extreme points. Let \( I \) be the index set of extreme points of \( \mathcal{P}_\rho \), and let us denote the set of extreme points of \( \mathcal{P}_\rho \) by \( \{ P^{(i)} \mid i \in I \} \).

The third constraint in (3.2) can be replaced by \( \max \{ \sum_{n \in \mathcal{N}_T} p_n^{(i)} (-Z_n^\top \Theta_n) \mid i \in I \} \leq 0 \).

Then the linear problem (3.2) turns into a linear program as follows:

\[
\max_{\Theta} \sum_{n \in \mathcal{N}_T} p_n^* Z_n^\top \Theta_n \\
\text{s.t.} \\
Z_0^\top \Theta_0 = 0 \\
Z_n^\top [\Theta_n - \Theta_{a(n)}] = 0 \\
\sum_{n \in \mathcal{N}_T} p_n^{(i)} (-Z_n^\top \Theta_n) \leq 0 \quad (i \in I) \\
\tag{3.3}
\]

Therefore, there is no \( \rho \)-arbitrage in the market if and only if the optimal value of (3.3) is zero.
The Lagrangian dual problem of (3.3) is

\[
\begin{align*}
\min_{(\lambda, \mu)} & \quad 0 \\
\text{s.t.} & \quad (p^*_n + \sum_{i \in I} \mu_i p^{(i)}_n - \lambda_n) Z_n = 0 \quad (n \in N_T) \\
& \quad \lambda_n Z_n = \sum_{m \in c(n)} \lambda_m Z_m \quad (n \in N_t, 0 \leq t \leq T - 1) \\
& \quad \mu_i \geq 0 \quad (i \in I).
\end{align*}
\]

By the assumption that \( z^0_n = 1 \) for all \( n \in \Omega \), any feasible solution of (3.4) satisfies

\[
\begin{align*}
\left\{ \begin{array}{l}
\lambda_n = p^*_n + \sum_{i \in I} \mu_i p^{(i)}_n \quad n \in N_T \\
\lambda_n = \sum_{m \in c(n)} \lambda_m \quad n \in N_t, \ t = 1, \ldots, T,
\end{array} \right.
\end{align*}
\]

implying that \( \lambda_0 = \sum_{n \in N_T} \lambda_n = \sum_{n \in N_T} (p^*_n + \sum_{i \in I} \mu_i p^{(i)}_n) = 1 + \sum_{i \in I} \mu_i > 0 \). Let \( q_n = \lambda_n / \lambda_0 \) for \( n \in \Omega \), then (3.4) is equivalent to the program below:

\[
\begin{align*}
\min_{(q, \mu)} & \quad 0 \\
\text{s.t.} & \quad q_n = \frac{p^*_n + \sum_{i \in I} \mu_i p^{(i)}_n}{1 + \sum_{i \in I} \mu_i} \quad (n \in N_T) \\
& \quad q_n Z_n^T = \sum_{m \in c(n)} q_m Z_m^T \quad (n \in N_t, t = 0, \ldots, T - 1) \\
& \quad q_n = \sum_{m \in c(n)} q_m \quad (n \in N_t, t = 0, \ldots, T - 1) \\
& \quad q_0 = 1 \\
& \quad \mu_i \geq 0 \quad (i \in I).
\end{align*}
\]

Since \( P_\rho \) is convex and \( P^* \in \text{int}(P_\rho) \), the first and fifth constraints imply \( Q \in \text{int}(P_\rho) \), where \( Q \) is the probability distribution modeled by attaching weights \( q_n \) to each node \( n \). Furthermore, the remaining constraints imply that \( Q \) is a martingale probability measure. We have that the problem (3.7) is feasible if and only if the set \( Q = \text{int}(P_\rho) \cap M \) is
According to the strong duality theorem of linear program (see [7] for example), the optimal value of (3.3) is zero if and only if (3.6) is feasible. Therefore we conclude that there is no $\rho$-arbitrage if and only if the set $Q = \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$ is nonempty.

We are now in the position to prove Theorem 3.6 by making use of the proof of Lemma 3.7.

**Proof of Theorem 3.6:**

(No $\rho$-arbitrage $\iff$ There exists a probability measure $Q^* \in \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$):

Suppose that there exists a probability measure $Q^* \in \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$. Since the set $\mathcal{P}_\rho$ is convex and closed, and both $Q^*$ and $P^*$ are interior points of $\mathcal{P}_\rho$, there exists some $Q' \in \mathcal{P}_\rho$ and $\mu' \geq 0$ such that $Q^* = \frac{P^* + \mu'Q'}{1 + \mu'}$.

Then $\lambda_n = p^*_n + \mu'q'_n = q'_n(1 + \mu')$ for all $n \in \Omega$ and $\mu = \mu'$ is a feasible solution of the linear program below:

$$
\begin{align*}
\min_{(\lambda, \mu)} & \quad 0 \\
\text{s.t.} & \quad (p^*_n + \mu'q'_n - \lambda_n)Z_n = 0 \quad (n \in \mathcal{N}_T) \\
& \quad \lambda_n Z_n = \sum_{m \in c(n)} \lambda_m Z_m \quad (n \in \mathcal{N}_t, 0 \leq t \leq T - 1) \\
& \quad \mu_i \geq 0 \quad (i \in I).
\end{align*}
$$

(3.7)

As shown in the proof of Lemma 3.7, the linear program (3.7) is the dual problem of the following linear program,

$$
\begin{align*}
\max_{(\Theta)} & \quad \sum_{n \in \mathcal{N}_T} p^*_n Z_n^\top \Theta_n \\
\text{s.t.} & \quad Z_0^\top \Theta_0 = 0 \\
& \quad Z_n^\top [\Theta_n - \Theta_{a(n)}] = 0 \quad (n \in \mathcal{N}_t, t \geq 1) \\
& \quad \sum_{n \in \mathcal{N}_T} q'_n (-Z_n^\top \Theta_n) \leq 0.
\end{align*}
$$

(3.8)
The strong duality theorem implies that the optimal value of (3.8) is zero.

Note that the optimal value of the problem (3.2) is either 0 or positively infinite, and is no larger than the optimal value of (3.8) in the sense of the extended acceptance set, and the optimal value of (3.2) is 0, which means the market is arbitrage free.

(No $\rho$-arbitrage $\implies$ There exists a probability measure $Q^* \in \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$):

Here we prove that if the optimal value of (3.2) is zero, then there exists a probability measure $Q^* \in \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$.

For simplicity, let us rewrite the program (3.2) as follows:

$$\max_{(x)} \ c^\top x$$
$$\text{s.t.} \quad a_\alpha^\top x \leq 0 \quad (\alpha \in A),$$

where $c, a_\alpha$ for all $\alpha \in A$ are given vectors generated by $P^*$ and $Z_n, n \in \Omega$. Set $A$ is an index set corresponding to $\mathcal{P}_\rho$.

Define $V := \{x \mid a_\alpha^\top x \leq 0, \ (\alpha \in A)\}$ and $V^* := \{y \mid y^\top x \leq 0, \ (x \in V)\}$. Obviously, the optimal value of (A.3) is zero if and only if $c \in V^*$.

Let $\mathcal{A} := \{a_\alpha \mid \alpha \in A\}$, and denote the minimal convex cone that contains $\mathcal{A}$ by $\text{cone}(\mathcal{A})$.

For any $\alpha \in A$, since the inequality $a_\alpha^\top x \leq 0$ holds for all $x \in V$, one has $\mathcal{A} \subseteq V^*$. Since $V^*$ is a convex cone, $\text{cone}(\mathcal{A}) \subseteq V^*$.

On the other hand, if there exists $y \in V^*$ such that $y \notin \text{cone}(\mathcal{A})$, then by the strong separation theorem, there exists a vector $d$, and a real number $l$ such that $d^\top y > l$ and $d^\top z < l$ for all $z \in \text{cone}(\mathcal{A})$.

By the fact that $0 \in \text{cone}(\mathcal{A})$, we have $l > 0$. By definition of $\text{cone}(\mathcal{A})$, for any $\lambda > 0$ and $z \in \text{cone}(\mathcal{A})$, $\lambda z$ remains in $\text{cone}(\mathcal{A})$, one has $d^\top z \leq 0$ for all $z \in \text{cone}(\mathcal{A})$, implying that $d \in V$. We conclude that there exists $d \in V$ such that $d^\top y > c > 0$ and $y \in V^*$, which contradicts to the definition of $V^*$. Therefore, there does not exist $y \in V^*$ such that $y \notin \text{cone}(\mathcal{A})$. Recalling that $\text{cone}(\mathcal{A}) \subseteq V^*$, we have $\text{cone}(\mathcal{A}) = V^*$. 

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The discussion above shows that if the optimal value of (3.9) is zero, then $c \in V^* = \text{cone}(A)$.

Let $\text{cov}(A)$ be the set of all nonnegative linear combinations of finitely many vectors of $A$. Note that $\text{cov}(A)$ is a convex cone that contains $A$, by the minimality of $\text{cone}(A)$, we also have $\text{cone}(A) \subseteq \text{cov}(A)$, which implies $c \in V^* \subseteq \text{cov}(A)$. From the definition of $\text{cov}(A)$, there exits a subset of $A$ with a finite number of elements $\{a_l \mid l \in L\}$ satisfies $c = \sum_{l \in L} \gamma_l a_l$ with all $\gamma_l \geq 0$.

Therefore, the optimal value of the following problem is zero.

$$\max_{(x)} \quad c^\top x = \sum_{l \in L} \gamma_l a_l^\top x \quad \text{s.t.} \quad a_l^\top x \leq 0 \quad (l \in L).$$

(3.10)

Similarly to the proof of Lemma 3.7, after applying the strong duality theorem to (3.10), we conclude that there exists a probability measure $Q^*$ in $\text{int}(P_\rho) \cap M$.

\[ \square \]

3.4 Geometrical explanation of the relation between the acceptance set and $Q$

In Section 3.3, we have shown the connection between the acceptance set and the no arbitrage condition mathematically. In this section, we are going to illustrate the relation between acceptance set and set $Q$ geometrically by taking an example of single period, two assets and two terminal nodes model. Let $z_n^i$ be the value of security $i$ ($i = 0, 1$) at each node $n$ ($n = 0, 1, 2$). Without loss of generality, we assume $z_n^0 = 1$ and $z_n^1 > z_n^2$. Let $\theta^i$ ($i = 1, 2$) be the amount of security $i$ contained in the portfolio. Let $Z_n = (z_n^0, z_n^1)$ for $n = 0, 1, 2$ and $\Theta = (\theta^0, \theta^1)$. Both $(z_n^0, z_n^1)$ and $(\theta^0, \theta^1)$ can be represented by points in $\mathbb{R}^2$ as shown in Figure 1. Denote the terminal net worth of the portfolio $\Theta$ by $X = (x_1, x_2)$, i.e., $x_1 = Z_1^\top \Theta$, $x_2 = Z_2^\top \Theta$. 

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3.4.1 Traditional arbitrage strategy

If the acceptance set $\mathcal{A}$ is $\{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0\}$, then the arbitrage defined by $\mathcal{A}$ is the traditional arbitrage. $\rho_{\mathcal{A}}((x_1, x_2))$ is then given by $\rho_{\mathcal{A}}((x_1, x_2)) = \min \{ c \mid c + x_1 \geq 0, c + x_2 \geq 0 \} = \max(-x_1, -x_2)$. It is easy to see that the set $\mathcal{P}_{\rho_{\mathcal{A}}}$ is equal to the set of all probability measures.

If $z_0^1 > z_1^1$, then a trading strategy $\Theta = (\theta^0, \theta^1)$ in the conical area DOE in Figure 3 would be possible.
satisfies

\[
\begin{cases}
Z_0^\top \Theta < 0 \\
Z_1^\top \Theta \geq 0; \ Z_2^\top \Theta \geq 0,
\end{cases}
\]  
(3.11)

and there exists an arbitrage opportunity in the market.

If \( z_0^1 = z_1^1 \), then a trading strategy \( \Theta = (\theta^0, \theta^1) \) on the line OD in Figure 3 satisfies

\[
\begin{cases}
Z_0^\top \Theta = 0 \\
Z_1^\top \Theta \geq 0; \ Z_2^\top \Theta \geq 0 \\
p_1^1 Z_1^\top \Theta + p_2^1 Z_2^\top \Theta = p_2^2 Z_2^\top \Theta > 0,
\end{cases}
\]  
(3.12)

and there exists an arbitrage opportunity also in this case.

Similarly, if \( z_0^1 \leq z_2^1 \), then there exists an arbitrage opportunity in the market. Therefore, the market is arbitrage free if and only if \( z_1^1 < z_0^1 < z_2^1 \), which implies \( z_0^1 = q_1 z_1^1 + q_2 z_2^1 \) for some \((q_1, q_2)\) such that \( q_1 > 0, q_2 > 0 \) and \( q_1 + q_2 = 1 \). This result is essentially the same as the first part of the fundamental theorem of asset pricing.
3.4.2 General type arbitrage

If the acceptance set is different from $L_+$, we may obtain a different condition for no arbitrage. Next we show how the acceptance set affects the condition for no arbitrage. Suppose that the acceptance set is defined as

$$A = \{(x_1, x_2) | a_1 x_1 + a_2 x_2 \leq 0; \ b_1 x_1 + b_2 x_2 \geq 0\}$$

for some positive numbers $a_1$, $a_2$, $b_1$, $b_2$ such that $a_1 + a_2 = 1$ and $b_1 + b_2 = 1$. Then $A$ satisfies the four properties listed in Section 3.2, and thus the risk measure associated with $A$ is a coherent risk measure. Without loss of generality, we assume $a_1 > b_1$.

![Figure 4: An example of the acceptance set properly containing $L_+$](image)

By definition, the risk measure $\rho_A((x_1, x_2)) = \min\{c|(x_1 + c, x_2 + c) \in A\}$ associated with $A = \{(x_1, x_2) | a_1 x_1 + a_2 x_2 \geq 0; \ b_1 x_1 + b_2 x_2 \geq 0\}$ is then simply given by

$$\rho_A((x_1, x_2)) = \begin{cases} -a_1 x_1 - a_2 x_2 & \text{if } x_1 \leq x_2 \\ -b_1 x_1 - b_2 x_2 & \text{otherwise.} \end{cases} \quad (3.13)$$

The set $P_{\rho_A}$ is then given by $P_{\rho_A} = \{(p_1, p_2) | 0 \leq p_1 \leq a_1; \ 0 \leq p_2 \leq b_2\}$.

If $z_0 > a_1 z_1 + a_2 z_2$, then a trading strategy $\Theta = (\theta^0, \theta^1)$ in the conical area DOE in Figure 5 satisfies

$$\begin{cases} Z_0^\top \Theta < 0 \\ a_1 Z_1^\top \Theta + a_2 Z_2^\top \Theta \geq 0; \ b_1 Z_1^\top \Theta + b_2 Z_2^\top \Theta \geq 0, \end{cases} \quad (3.14)$$
and there exists a $\rho_A$-arbitrage opportunity in the market.

Figure 5: Existence of a $\rho_A$-arbitrage

If $z_1^0 = a_1 z_1 + a_2 z_2$, then a trading strategy $\Theta = (\theta^0, \theta^1)$ on the line OE in Figure 5 satisfies

$$
\begin{cases}
Z_0^\top \Theta = 0 \\
a_1 Z_1^\top \Theta + a_2 Z_2^\top \Theta \geq 0; \quad b_1 Z_1^\top \Theta + b_2 Z_2^\top \Theta \geq 0 \\
p_1^* Z_1^\top \Theta + p_2^* Z_2^\top \Theta > 0
\end{cases}
$$

if $(p_1^*, p_2^*) \in \rho_A$, implying a $\rho_A$-arbitrage in the market.

Figure 6: No existence of the $\rho_A$-arbitrage
Similarly, if \( z_1 \leq b_1 z_1 + b_2 z_2 \), there exists a \( \rho \)-arbitrage opportunity in the market. Therefore, the market is \( \rho \)-arbitrage free if and only if the point \( A \) is located between the points \( M \) and \( N \) as depicted in Figure 6, which means \( z_0^1 = q_1 z_1^1 + q_2 z_1^2 \) holds for some \( (q_1, q_2) \) such that \( 0 < q_1 < a_1, \ 0 < q_2 < a_2 \) and \( q_1 + q_2 = 1 \). This result is essentially equivalent to Theorem 3.6 we have proved.

4 Pricing the European style contingent claims

European style contingent claims are a class of contingent claims which provide the holder with the right to receive payoff at maturity date \( T \). An example of the European style contingent claims is European call option on a specific security, which gives the holder the right to buy the security for price \( K \) at maturity date \( T \). The value of the option at \( T \) is then given by \( \max\{0, S_T - K\} \) where \( S_T \) denotes the security price at the maturity date \( T \).

In this chapter we consider the pricing problem of European style contingent claims. To solve this pricing problem, we first consider the problem of hedging the risk arising from the European style contingent claims.

Definition 4.1 (\( \rho \)-hedging) Let \( C_n \) denote the value of a contingent claim \( C \) at each terminal node \( n \in \mathcal{N}_T \). A self-financing trading strategy \( \Theta \) is the \( \rho \)-hedging strategy of \( C \) if

1. the expected value of the portfolio that contains \( \theta_n^j \) units of security \( j, j = 0, \ldots, J \) and \( -1 \) unit of claim is nonnegative, i.e., \( \sum_{n \in \mathcal{N}_T} p_n^*(Z_n^\top \Theta_n - C_n) \geq 0 \), and
2. the future wealth of the portfolio is acceptable with respect to the coherent risk measure \( \rho \), i.e., \( \rho(Z_T^\top \Theta_T - C) = \max\{\sum_{n \in \mathcal{N}_T} p_n(-Z_n^\top \Theta_n + C_n) \mid P \in \mathcal{P}_\rho\} \leq 0 \).

Remark 4.2 In the no \( \rho \)-arbitrage market, the cost to \( \rho \)-hedge a security \( C \) must be greater than the initial value of the security \( C \). In other words, if \( \Theta \) can \( \rho \)-hedge \( C \), then \( Z_0^\top \Theta_0 \geq C_0 \), where \( C_0 \) is the initial value of \( C \).
Remark 4.3 If \( P^* \in \mathcal{P}_\rho \), then \( \rho(Z_T^\top \Theta_T - C) \leq 0 \) implies \( \sum_{n \in \mathcal{N}_T} p_n^*( -Z_n^\top \Theta_n + C_n ) \leq 0 \).

4.1 Computation of the upper bound of European style contingent claims

In this chapter we consider the pricing problem of European style contingent claim \( Y \) with maturity date \( T \).

Let \( Y_n \) denote the future payment from the seller of the European style contingent claim \( Y \) at each terminal node \( n \in \mathcal{N}_T \). The upper bound of the price of the claim is then the minimum cost for the seller to \( \rho \)-hedge this claim, which is mathematically given by the optimal value of the linear program:

\[
\begin{align*}
\min_{\Theta} & \quad Z_0^\top \Theta_0 \\
\text{s.t.} & \quad Z_n^\top [\Theta_n - \Theta_{a(n)}] = 0 & (n \in \mathcal{N}_t, \ t = 1, \ldots, T) \\
& \quad \max \{ E^P(-Z_T^\top \Theta_T + \delta_T Y_T) \mid P \in \mathcal{P}_\rho \} \leq 0 \\
& \quad E^{P^*}(Z_T^\top \Theta_T - \delta_T Y_T) \geq 0.
\end{align*}
\]  

(4.1)

If \( P^* \in \text{int}(\mathcal{P}_\rho) \), then \( \max \{ E^P(-Z_T^\top \Theta_T + \delta_T Y_T) \mid P \in \mathcal{P}_\rho \} \leq 0 \) implies \( E^{P^*}(Z_T^\top \Theta_T - \delta_T Y_T) \geq 0 \), which means the third constraint of (4.1) can be omitted, hence the program (4.1) is equivalent to the following program.

\[
\begin{align*}
\min_{\Theta} & \quad Z_0^\top \Theta_0 \\
\text{s.t.} & \quad Z_n^\top [\Theta_n - \Theta_{a(n)}] = 0 & (n \in \mathcal{N}_t, \ t = 1, \ldots, T) \\
& \quad \max \{ E^P(-Z_T^\top \Theta_T + \delta_T Y_T) \mid P \in \mathcal{P}_\rho \} \leq 0.
\end{align*}
\]  

(4.2)

Furthermore, if \( \mathcal{P}_\rho \) is a polytope, the second constraint of (4.2) can be replaced by \( \max \{ E^{P(i)}(-Z_T^\top \Theta_T + \delta_T Y_T) \mid i \in I \} \leq 0 \) where \( I \) is the index set of the extreme points.
\( P^{(i)} \) of \( \mathcal{P}_\rho \), and (4.2) is then equivalent to

\[
\begin{align*}
\min_{(\Theta)} & \quad Z_0^\top \Theta_0 \\
n\text{s.t.} & \quad Z_n^\top [\Theta_n - \Theta_{a(n)}] = 0 \quad (n \in \mathcal{N}_t, \ t = 1, \ldots, T) \\
& \quad E^{P^{(i)}} (-Z_T^\top \Theta_T + \delta_T \gamma_T) \leq 0 \quad (i \in I).
\end{align*}
\]

(4.3)

Similarly to (3.2), the first step to solve problem (4.3) is to obtain the dual problem. The Lagrangian function of (4.3) is given by

\[
L(\Theta; q, \mu) = Z_0^\top \Theta_0 + \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n Z_n^\top [\Theta_n - \Theta_{a(n)}] + \sum_{i \in I} \mu_i \sum_{n \in \mathcal{N}_T} p_n^{(i)} (-Z_n^\top \Theta_n + \delta_n \gamma_n) = \sum_{n \in \mathcal{N}_T} \left( \sum_{i \in I} \mu_i p_n^{(i)} \right) \delta_n \gamma_n + \left( Z_0^\top - \sum_{m \in c(0)} q_m Z_m^\top \right) \Theta_0 + \sum_{n \in \mathcal{N}_T} \left( q_n - \sum_{i \in I} \mu_i p_n^{(i)} \right) Z_n^\top \Theta_n.
\]

(4.4)

By putting \( q_0 = 1 \), the Lagrangian dual problem of (4.3) is

\[
\begin{align*}
\max_{(q, \mu)} & \quad \sum_{n \in \mathcal{N}_T} \left( \sum_{i \in I} \mu_i p_n^{(i)} \right) \delta_n \gamma_n \\
n\text{s.t.} & \quad q_0 = 1 \\
& \quad q_n Z_n = \sum_{m \in c(n)} q_m Z_m \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T - 1) \quad (4.5) \\
& \quad q_n Z_n = \sum_{i \in I} \mu_i p_n^{(i)} Z_n \quad (n \in \mathcal{N}_T) \\
& \quad \mu \geq 0.
\end{align*}
\]

It should be noted that since \( z_n^0 \equiv 1 \) for every \( n \in \Omega \), any feasible solution \((q, \mu)\) of (4.5) satisfies the following system:

\[
\begin{align*}
\sum_{n \in \mathcal{N}_t} q_n = q_0 = 1 \quad (t = 0, \ldots, T - 1) \\
q_n = \sum_{i \in I} \mu_i p_n^{(i)} \quad (n \in \mathcal{N}_T) \quad (4.6) \\
\mu \geq 0.
\end{align*}
\]
Accordingly, the second constraint of (4.5) implies that the probability measure \( Q \) modeled by \( q_n \) is a martingale probability measure. Furthermore, taking the summation of the third constraint of (4.6) over \( n \in \mathcal{N}_T \), we have \( 1 = \sum_{n \in \mathcal{N}_T} q_n = \sum_{n \in \mathcal{N}_T} (\sum_{i \in I} \mu_i p_n^{(i)}) = \sum_{i \in I} \mu_i \). The non-negativity of all \( \mu_i \) with \( i \in I \) implies that \( Q \) is a convex combination of all \( P^{(i)}, i \in I \). Therefore, we have \( Q \in \mathcal{P}_\rho \) by the convexity of \( \mathcal{P}_\rho \).

The problem (4.5) is finally equivalent to

\[
\begin{align*}
\max_{Q} & \quad E^Q(\delta_T Y_T) \\
\text{s.t.} & \quad Q \in \mathcal{P}_\rho \cap \overline{\mathcal{M}} = \overline{Q},
\end{align*}
\]

where \( Q = \text{int}(\mathcal{P}_\rho) \cap \mathcal{M} \), and \( \overline{Q} \) is the closure of \( Q \).

According to the strong duality theorem and Theorem 4.5 in the previous section, if the market is \( \rho \)-arbitrage free, then the problem (4.7) is feasible, and the optimal value of (4.1) is equal to that of program (4.7) which is \( \max_{Q \in \overline{Q}} E^Q(\delta_T Y_T) \). In other words, the no \( \rho \)-arbitrage price \( Y_0 \) of security \( Y \) satisfies

\[
Y_0 \leq \max_{Q \in \overline{Q}} E^Q(\delta_T Y_T).
\]

### 4.2 Computation of the lower bound of European style contingent claims

The buyer pays \( Y_0 \) in return for a promise of payments \( Y_n \) at each terminal node \( n \in \mathcal{N}_T \), and the exposure of buyer is then \(-Y_n\). Under the assumption of no \( \rho \)-arbitrage, the cost for \( \rho \)-hedging this position is no less than initial value of this position. Then the buyer’s
The problem may be modeled as the following linear program:

\[
\begin{align*}
\min_{\Theta} & \quad Z_0^T \Theta_0 \\
\text{s.t.} & \quad Z_n^T (\Theta_n - \Theta_{a(n)}) = 0, \quad (n \in \mathcal{N}_t, \ t = 1, \ldots, T) \\
& \quad \max \{ E^P(-Z_T^T \Theta_T - \delta_T Y_T) \mid P \in \mathcal{P}_\rho \} \leq 0 \\
& \quad \sum_{n \in \mathcal{N}_T} p^*_n (Z_n^T \Theta_n + \delta_n Y_n) \geq 0.
\end{align*}
\]

The optimal value of (4.8) gives the upper bound of initial value $-Y_0$.

Similarly to (4.1), in the no $\rho$-arbitrage market, $-Y_0$ satisfies

\[
-Y_0 \leq \max_{Q \in \mathcal{Q}} \sum_{n \in \mathcal{N}_T} q_n (-\delta_n Y_n) = -\min_{Q \in \mathcal{Q}} \sum_{n \in \mathcal{N}_T} q_n \delta_n Y_n,
\]

if $\mathcal{P}_\rho$ is a polytope and $P^* \in \text{int}(\mathcal{P}_\rho)$. Consequently, we have the lower bound of the price $Y_0 \geq \min_{Q \in \mathcal{Q}} E^Q(\delta_T Y_T)$.

### 4.3 In the case of $\mathcal{P}_\rho$ is not a polytope

The Section 3.3 has shown that the market with $J + 1$ tradable assets is $\rho$-arbitrage free if and only if there exists a probability measure $Q$ satisfying

\[
\begin{align*}
q_n z_n^j = \sum_{m \in c(m)} q_m z_m^j, \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T - 1; \ j = 0, \ldots, J) \\
Q \in \text{int}(\mathcal{P}_\rho),
\end{align*}
\]

when $\rho$ is any coherent risk measure.

Let us consider an extended market with the $J + 1$ tradable assets and an additional European style claim $Y$, since the result of Section 3.3 is independent of the number $J$, one has that the extended market remains arbitrage free if and only if there exists a
probability measure $Q$ such that

\[
\begin{aligned}
q_n z_n^j &= \sum_{m \in c(m)} q_n z_m^j \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T - 1; \ j = 0, \ldots, J) \\
q_n Y_n &= \sum_{m \in c(m)} q_m Y_m \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T - 1) \\
Q &\in \text{int}(\mathcal{P}_\rho),
\end{aligned}
\]

which means the price $Y_0$ of $Y$ satisfies $Y_0 = E^Q(Y_T)$ with $Q \in \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$ where $\mathcal{M} = \{ Q \mid Q > 0, \ q_n z_n^j = \sum_{m \in c(m)} q_m z_m^j \text{ for all } n \in \mathcal{N}_t, \ t = 0, \ldots, T - 1 \text{ and } j = 0, \ldots, J \}$. Therefore

\[
\inf_{Q \in \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}} E^Q(Y_T) \leq Y_0 \leq \sup_{Q \in \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}} E^Q(Y_T).
\]

Since both the sets $\mathcal{P}_\rho$ and $\mathcal{M}$ are convex and closed, and the function $E^Q(Y_T)$ is continuous in $Q$, we have

\[
\min_{Q \in \mathcal{P}_\rho \cap \mathcal{M}} E^Q(Y_T) \leq Y_0 \leq \max_{Q \in \mathcal{P}_\rho \cap \mathcal{M}} E^Q(Y_T).
\]

\[
\square
\]

4.4 No $\rho$-arbitrage interval of $Y_0$

**Theorem 4.4** Let $\rho$ be any coherent risk measure with which $\mathcal{P}_\rho$ is a polytope and suppose $P^* \in \text{int}(\mathcal{P}_\rho)$, and the market is no $\rho$-arbitrage market. The price $Y_0$ of European style contingent claim $Y$ satisfies

\[
\begin{aligned}
\min_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n \delta_n Y_n &= Y_0 = \max_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n \delta_n Y_n \quad \text{if } \max = \min \\
\min_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n \delta_n Y_n < Y_0 < \max_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n \delta_n Y_n \quad \text{otherwise},
\end{aligned}
\]

where $Y_n$ is the payment to the buyer at any state $n \in \mathcal{N}_T$.

**Proof:** The discussions in the previous sections of this chapter show that the upper and lower bounds for $Y_0$ are given by $\min_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n \delta_n Y_n$ and $\max_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n \delta_n Y_n$, respectively.
Therefore, all we need to prove here is the equivalence that

\[
\min_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n = Y_0 \iff \max_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n = Y_0
\]

in the arbitrage free market.

[Case 1. \(Y_0 = \max_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n\): Let \(\Theta^*\) be an optimal solution of (4.1). If \(Y_0 = \max_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n\) is a no \(\rho\)-arbitrage price, then \(\rho(Z_T^T \Theta_T^* - \delta_T Y_T) = 0\) and \(E^P(Z_T^T \Theta_T^* - \delta_T Y_T) = 0\) imply that \(E^P(Z_T^T \Theta_T^* - \delta_T Y_T) = 0\) for all \(P \in \mathcal{P}_\rho\). Therefore, \(-\Theta^*\) is a feasible solution of (4.8). \(-Z_0^T \Theta_0^*\) is no smaller than the optimal value of (4.8)

\[-\min_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n,\]

which means \(\min_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n \geq Z_0^T \Theta_0^*\).

On the other hand, \(\Theta^*\) is an optimal solution of (4.1), we have \(\max_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n = Z_0^T \Theta_0^*\). Therefore,

\[Y_0 = Z_0^T \Theta_0^* = \min_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n = \max_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n.\]

[Case 2. \(Y_0 = \min_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n\): \(Y_0 = \min_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n\) also leads to \(Y_0 = \max_{Q \in \bar{Q}} \frac{1}{\delta_0} \sum_{n \in N_T} q_n \delta_n Y_n\).]

\[\square\]

5 Pricing the American style contingent claims

In this chapter, we consider another class of contingent claims which are referred to as American style contingent claims. The difference between the European and American claims is that the holder of the American claim has the right to decide to exercise the contract at any time before the maturity date \(T\).
5.1 Computation of the lower bound of American style contingent claims

Let $Y_n$ denote the payoff from the seller to the buyer when the contingent claim is exercised at node $n \in \mathcal{N}_t$, $t = 0, \ldots, T$. The exposure of buyer is $-Y_n$ if the buyer decides to exercise the claim at node $n$, we also construct a problem of minimizing the cost to $\rho$ hedging the claim. If the buyer decides to exercise at state $n$, the total amount he can invest next time will be $Z_n^\top \Theta_n + \delta_n Y_n$.

The buyer’s problem is given as

$$\min_{e} \min_{(\Theta, x)} Z_0^\top \Theta_0$$

subject to

$$Z_n^\top [\Theta_n - \Theta_{a(n)}] = e_n \delta_n Y_n \quad \quad (n \in \mathcal{N}_t, \ t = 1, \ldots, T)$$

$$\rho(Z_T^\top \Theta_T) = \max \{ E^P(-Z_T^\top \Theta_T) \mid P \in \mathcal{P}_\rho \} \leq 0$$

$$\sum_{n \in \mathcal{N}_T} p_n^* (Z_n^\top \Theta_n) \geq 0$$

$$\sum_{m \in A(n)} e_m \leq 1 \quad \quad (n \in \mathcal{N}_T)$$

$$e_n \in \{0, 1\} \quad \quad (n \in \Omega),$$

where $A(n)$ denotes the history path of any node $n \in \mathcal{N}_T$ including $n$ itself, i.e., $A(n) = \{ n_0, n_1, \ldots, n_t, \ldots, n_T \mid n_0 = 0, n_T = n, and n_t = a(n_{t+1})\}$ for all $t = 0, \ldots, T - 1$.

For example, in a two periods model ($T = 2$), if for one terminal node $n_2 \in \mathcal{N}_T$, one has $n_1 \in \mathcal{N}_1$ is the parent node of $n_2$ and $n_0 = 0$ is the parent node of $n_1$, then $A(n_2)$ is given by $\{ n_0, n_1, n_2 \}$.

Keeping $e$ fixed in (5.1) and minimizing with respect to $\Theta$ reduces to the pricing problem of European style claim with payoff $e_n Y_n$ at each node $n \in \Omega$, the optimal value of (5.1) for fixed $e$ is then

$$\max_{Q \in \mathcal{Q}} \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} (-q_n e_n \delta_n Y_n)$$
if $P_\rho$ is a polytope and $P^* \in (P_\rho)$. The optimal value of (5.1) is then equal to

$$
\min_{e} \max_{Q \in \mathcal{Q}} \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} (-q_n e_n \delta_n Y_n) = -\max_{e} \min_{Q \in \mathcal{Q}} \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} q_n e_n \delta_n Y_n.
$$

which reduces to $-Y_0 \leq -\max_{e} \min_{Q \in \mathcal{Q}} \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} q_n e_n \delta_n Y_n$. Therefore, we have

$$
Y_0 \geq \max_{e} \min_{Q \in \mathcal{Q}} \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} q_n e_n \delta_n Y_n.
$$

Instead of the variable $e$, it is more common to describe exercise strategies for an American style claim through stopping times which are functions $\tau : \Omega \rightarrow \{0, 1, \ldots, T\} \cup \{+\infty\}$ such that $\{n \in \Omega \mid \tau(n) = t\} \in \mathcal{F}_t$, for each $t = 0, 1, \ldots, T$. The relation $e_n = 1, n \in \mathcal{N}_t \iff \tau(n) = t$ defines a correspondence between stopping times and exercise strategy $e$.

**Theorem 5.1** For any coherent risk measure $\rho$, if $P_\rho$ is a polytope and $P^* \in \text{int}(P_\rho)$, then in $\rho$-arbitrage free market, the price $F_0$ of the American style claim satisfies

$$
Y_0 \geq \frac{1}{\delta_0} \max_{\tau \in \Gamma} \min_{Q \in \mathcal{Q}} E^Q(\delta_\tau Y_{\tau}).
$$

### 5.2 Computation of the upper bound of American style contingent claims

Since the holder of the claim has the right to exercise the claim, the seller has to hedge any possible payments before or at the maturity date. The optimal value of the following program is the minimum value to $\rho$-hedge the claim.

$$
\begin{align*}
\min_{(\Theta)} \quad & Z_0^\top \Theta_0 \\
\text{s.t.} \quad & Z_n^\top [\Theta_n - \Theta_{a(n)}] = 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (n \in \mathcal{N}_t, \ t = 1, \ldots, T) \\
& \max \{ E^P(-Z_t^\top \Theta_t + \delta_t Y_t) \mid P \in \mathcal{P}_\rho \} \leq 0 \quad (t = 0, \ldots, T).
\end{align*}
$$
If an optimal solution $\Theta^*$ of (5.2) satisfies $Z_n^\top \Theta^*_n < \delta_n Y_n$ for some $n^* \in \Omega \setminus \mathcal{N}_T$, then a trading strategy with $\theta_n^{(j)}$ units of security $j$ and $-1$ unit of claim for all $n \in \Omega$ is a $\rho$-arbitrage trading strategy since the value of this portfolio at node $n^*$ is $Z_n^\top \Theta_n^* - \delta_n Y_n < 0$, and at the maturity date $T$, the risk of portfolio $\max\{ E^P(-Z_T^\top \Theta_T^* + \delta_T Y_T) \mid P \in \mathcal{P}_\rho \}$ is non-positive, which leads to $E^P(Z_T^\top \Theta_T^* - \delta_T Y_T) \geq 0$ if $P^* \in \text{int}(\mathcal{P}_\rho)$.

The discussion above can be concluded as that:

**Theorem 5.2** For any coherent risk measure $\rho$ with $P^* \in \text{int}(\mathcal{P}_\rho)$, under the assumption of no $\rho$-arbitrage, the two problems (5.2) and (5.3) are equivalent.

\[
\begin{align*}
\min_{\Theta} & \quad Z_0^\top \Theta_0 \\
\text{s.t.} & \quad Z_n^\top [\Theta_n - \Theta_{\alpha(n)}] = 0 \quad (n \in \mathcal{N}_t, \ t = 1, \ldots, T) \\
& \quad Z_n^\top \Theta_n \geq \delta_n Y_n \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T - 1) \\
& \quad \max\{ E^P(-Z_T^\top \Theta_T + \delta_T Y_T) \mid P \in \mathcal{P}_\rho \} \leq 0 
\end{align*}
\]

Calculating the upper bound of $Y_0$ then turns into the problem of solving the stochastic program (5.3)

**Theorem 5.3** Let $\rho$ be any coherent risk measure with $P^* \in \text{int}(\mathcal{P}_\rho)$ and $\mathcal{P}_\rho$ is a polytope. If the market is $\rho$-arbitrage free, then the minimum cost for the seller to $\rho$ hedging the cash flow of American style claim $Y$ is given by

\[
\frac{1}{\delta_0} \max_{\tau \in \Gamma} \max_{\mathcal{Q} \in \mathcal{Q}} E^\mathcal{Q}(\delta_\tau Y_\tau)
\]

where $\mathcal{Q} = \text{int}(\mathcal{P}_\rho) \cap \mathcal{M}$.

**Proof:** We first replace the second constraint of (5.3) by $\max\{ E^{P(i)}(-Z_T^\top \Theta_T + \delta_T Y_T) \mid i \in I \} \leq 0$, where $I$ is the index set of the extreme points of $\mathcal{P}_\rho$. The program (5.3) turns
\[
\begin{align*}
\min_{\Theta} & \quad Z_0^T \Theta_0 \\
\text{s.t.} & \quad Z_n^T \Theta_n = 0 \quad (n \in \mathcal{N}_t, \ t = 1, \ldots, T) \\
& \quad Z_n^T \Theta_n \geq \delta_n Y_n \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T - 1) \\
& \quad E^{(i)} (-Z_T^T \Theta_T + \delta_T Y_T) \leq 0 \quad (i \in I)
\end{align*}
\]

The dual problem of (5.4) is then

\[
\begin{align*}
\max_{(\lambda; \mu; \tau)} & \quad \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} \tau_n \delta_n Y_n \\
\text{s.t.} & \quad \lambda_0 = 1 \\
& \quad \lambda_n = \tau_n = \sum_{i \in I} \mu_i p_n^{(i)} \quad (n \in \mathcal{N}_T) \\
& \quad \sum_{m \in c(n)} \lambda_m Z_m^T = (\lambda_n - \tau_n) Z_n^T \quad (n \in \mathcal{N}_t; \ t = 0, \ldots, T - 1) \\
& \quad \tau_n \geq 0 \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T), \\
& \quad \mu_i \geq 0 \quad (i \in I).
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\max_{(\lambda; \mu; \tau)} & \quad \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} \tau_n \delta_n Y_n \\
\text{s.t.} & \quad \lambda_0 = 1 \\
& \quad \tau_n = \lambda_n = \sum_{i \in I} \mu_i p_n^{(i)} \quad (n \in \mathcal{N}_T) \\
& \quad \sum_{m \in c(n)} \lambda_m Z_m^T = (\lambda_n - \tau_n) Z_n^T \quad (n \in \mathcal{N}_t; \ t = 0, \ldots, T - 1) \\
& \quad \frac{\lambda_n}{\sum_{n \in \mathcal{N}_t} \lambda_n} = \sum_{i \in I} \mu_i p_n^{(i)} \quad (n \in \mathcal{N}_T) \\
& \quad \tau_n \geq 0 \quad (n \in \mathcal{N}_t, \ t = 0, \ldots, T) \\
& \quad \mu_i \geq 0 \quad (i \in I).
\end{align*}
\]

Let \( p_n = \sum_{i \in I} \frac{\mu_i p_n^{(i)}}{\sum_{i \in I} \mu_i} \) for all \( n \in \mathcal{N}_T \), \( p_n \) is then a convex combination of \( p_n^{(i)}, \ i \in I \). The
convexity of $\mathcal{P}_\rho$ implies that probability measure $P$ modeled by $p_n$ belongs to set $\mathcal{P}_\rho$. The problem (5.6) can be rewritten as

$$\begin{aligned}
\max_{(\lambda, \tau, P)} & \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} \tau_n \delta_n Y_n \\
\text{s.t.} & \quad \lambda_0 = 1 \\
& \quad \tau_n = \lambda_n \quad (n \in \mathcal{N}_T) \\
& \quad \sum_{m \in c(n)} \frac{\lambda_m Z_m^\top}{\lambda_n - \tau_n} = \frac{\sum_{m \in c(n)} \lambda_m Z_m^\top}{\sum_{m' \in c(n)} \lambda_{m'}} \quad (n \in \mathcal{N}_i; \ t = 0, \ldots, T - 1) \quad (5.7) \\
& \quad \sum_{n \in \mathcal{N}_T} \frac{\lambda_n}{\lambda_n} = p_n \quad (n \in \mathcal{N}_T) \\
& \quad \tau_n \geq 0 \quad (n \in \mathcal{N}_i, \ t = 0, \ldots, T). 
\end{aligned}$$

In the case of $\lambda_n > \tau_n$ for all $n \in \mathcal{N}_i, \ t = 0, \ldots, T - 1$, we have

$$\begin{aligned}
\max_{(\lambda, \tau, P)} & \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} \tau_n \delta_n Y_n \\
\text{s.t.} & \quad \lambda_0 = 1 \\
& \quad \tau_n = \lambda_n \quad (n \in \mathcal{N}_T) \\
& \quad \sum_{m \in c(n)} \frac{\lambda_m}{\lambda_n - \tau_n} Z_m^\top = \sum_{m \in c(n)} \frac{\lambda_m}{\lambda_n} Z_n^\top = Z_n^\top \quad (n \in \mathcal{N}_i; \ t = 0, \ldots, T - 1) \quad (5.8) \\
& \quad \sum_{n' \in \mathcal{N}_T} \frac{\lambda_n}{\lambda_n} = p_n \quad (n \in \mathcal{N}_T) \\
& \quad \tau_n \geq 0 \quad (n \in \mathcal{N}_i, \ t = 0, \ldots, T - 1). 
\end{aligned}$$

The second and third constraints imply that (5.8) is feasible if and only if there exists an equivalent martingale probability measure $Q$ such that

$$\frac{\lambda_m}{\lambda_n - \tau_n} = \frac{q_m}{q_n}, \text{ which leads to } \frac{\lambda_m}{q_m} = \frac{\lambda_n - \tau_n}{q_n}$$

and $Q = P$. Making the change of variables

$$f_n = \frac{\lambda_n}{q_n}; \ e_n = \frac{\tau_n}{q_n}, \text{ for each } n \in \Omega,$$
we can express (5.8) as

\[
\max_{(f,e;Q)} \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} q_n e_n \delta_n Y_n
\]

s.t. \( f_0 = 1 \)

\( f_n = f_{a(n)} - e_{a(n)} \) \( (n \in \mathcal{N}_t, t = 1, \ldots, T) \)

\( f_n = e_n \) \( (n \in \mathcal{N}_T) \)

\( f_n > e_n \geq 0 \) \( (n \in \Omega \setminus \mathcal{N}_T) \)

\( Q \in \bar{Q} \) (5.9)

By taking the summation of the second constraint over each \( m \in A(n) \), one has

\[
\sum_{m \in A(n)} f_m - f_0 = \sum_{m \in A(n)} f_m - f_n - \sum_{m \in A(n)} e_m + e_n \text{ which reduces to } \sum_{m \in A(n)} e_m = 1
\]

for all \( n \in \mathcal{N}_T \), where \( A(n) \) is the history path of \( n \) including \( n \) itself.

For any fixed \( e \), the optimal value of (5.9) is

\[
\max_{Q \in \bar{Q}} \max_{\tau \in \Gamma} E^Q(\delta_{\tau} e_n Y_{\tau}).
\]

And for any fixed \( Q \), \( \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} q_n e_n \delta_n Y_n \) is linear in \( e \), its optimal value is then attained at an extreme point which satisfies \( e_m \in \{0, 1\} \) for each \( m \in A(n) \) and \( \sum_{m \in A(n)} e_m = 1 \).

Thus the optimal value of (5.9) is

\[
\max_{\tau \in \Gamma} \max_{Q \in \bar{Q}} E^Q(\delta_{\tau} Y_{\tau}).
\]

Therefore, in the no \( \rho \)-arbitrage market, the price \( Y_0 \) of American style claim \( Y \) satisfies

\[
Y_0 \leq \frac{1}{\delta_0} \max_{\tau \in \Gamma} \max_{Q \in \bar{Q}} E^Q(\delta_{\tau} Y_{\tau}).
\]

\( \square \)
Theorem 5.4 For any coherent risk measure $\rho$ with $\mathcal{P}_\rho$ be a polytope and $P^* \in \text{int}(\mathcal{P}_\rho)$, if the market is a no $\rho$-arbitrage market, then the price $Y_0$ of American style claim $Y$ satisfies:

$$\begin{cases}
\frac{1}{\delta_0} \max \min_{\tau \in \Gamma} E^Q(\delta_\tau Y_\tau) = Y_0 = \frac{1}{\delta_0} \max \max_{Q \in \bar{Q}} E^Q(\delta_\tau Y_\tau) & \text{if max = min} \\
\frac{1}{\delta_0} \max \min_{\tau \in \Gamma} E^Q(\delta_\tau Y_\tau) < Y_0 < \frac{1}{\delta_0} \max \max_{Q \in \bar{Q}} E^Q(\delta_\tau Y_\tau) & \text{otherwise.}
\end{cases}$$

6 Numerical Example

In the preceding parts of this thesis, we have proposed a method on contingent claim hedging and pricing when the tradable assets’ price process follows a stochastic process represented by a scenario tree process. In this chapter, we implement our methodology in a practical setting. The first step is to generate a scenario tree for the price process.

6.1 Scenario tree generation

If the price process is defined on a finite discrete space, the generation of the tree is straightforward, as it can be done manually. However, in all other cases, constructing the tree manually is practically difficult.

Hoyland and Wallace [11] introduced a methodology for generating a scenario tree when given several statistical properties of the process. In this section, we illustrate their scenario generation method as well as our pricing method by way of a multiple-period example.

Hoyland and Wallace [11] introduced their method through a two periods example, they supposed that there are three tradable assets in the market: the first one is a risk-free bond with current price 1; the second one is a domestic stock index with current price 100; and the third one is an international stock index with current price 200. The problem is to calculate the price of a European call option on domestic stock index with strike price 100 and maturity date $T = 2$. 
Table 1 summarizes the correlations among these three assets, estimated by empirical data, and Table 2 shows the specification of market expectations.

<table>
<thead>
<tr>
<th>Correlations</th>
<th>Risk-free bond</th>
<th>Domestic index</th>
<th>International index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk free bond</td>
<td>1.0</td>
<td>-0.2</td>
<td>-0.1</td>
</tr>
<tr>
<td>Domestic index</td>
<td>-0.2</td>
<td>1.0</td>
<td>0.6</td>
</tr>
<tr>
<td>International index</td>
<td>-0.1</td>
<td>0.6</td>
<td>1.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specification</th>
<th>Distribution Property</th>
<th>End of Period 1</th>
<th>End of Period2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk free bond</td>
<td>expected value</td>
<td>4.33%</td>
<td>state dependent</td>
</tr>
<tr>
<td></td>
<td>standard deviation</td>
<td>0.94%</td>
<td>state dependent</td>
</tr>
<tr>
<td></td>
<td>skewness</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>kurtosis</td>
<td>2.62</td>
<td>2.62</td>
</tr>
<tr>
<td></td>
<td>worst-case event</td>
<td>6.68%</td>
<td>state dependent</td>
</tr>
<tr>
<td>Domestic index</td>
<td>expected value</td>
<td>7.61%</td>
<td>state dependent</td>
</tr>
<tr>
<td></td>
<td>standard deviation</td>
<td>13.38%</td>
<td>state dependent</td>
</tr>
<tr>
<td></td>
<td>skewness</td>
<td>-0.75</td>
<td>-0.75</td>
</tr>
<tr>
<td></td>
<td>kurtosis</td>
<td>2.93</td>
<td>2.93</td>
</tr>
<tr>
<td></td>
<td>worst-case event</td>
<td>-25.84%</td>
<td>state dependent</td>
</tr>
<tr>
<td>International index</td>
<td>expected value</td>
<td>8.09%</td>
<td>state dependent</td>
</tr>
<tr>
<td></td>
<td>standard deviation</td>
<td>15.70%</td>
<td>state dependent</td>
</tr>
<tr>
<td></td>
<td>skewness</td>
<td>-0.74</td>
<td>-0.74</td>
</tr>
<tr>
<td></td>
<td>kurtosis</td>
<td>2.97</td>
<td>2.97</td>
</tr>
<tr>
<td></td>
<td>worst-case event</td>
<td>-31.16%</td>
<td>state dependent</td>
</tr>
</tbody>
</table>

In this example, there are 33 specified statistical properties. Let $F$ be the set of all these specified statistical properties, for each statistical property $i$, $(i = 1, 2, \ldots, 33)$, and let $f_{VAL_i}$ be the specified statistical properties value. We first generate a two periods symmetrical tree that matches these 33 specified statistical properties. A symmetrical tree is a scenario tree with the constant number of branches at each node $n \in \mathcal{N}_t$, $t = 0, \ldots, T - 1$. A symmetrical tree is specified by the number of branches for all conditional distributions, the probability density of each node and the value of each asset at each node. Let $N$ denote the number of branches for each conditional distribution, $p_n^*$ the
probability of each node $n$, and $s_{jn}^j$ the price of each asset $j$ ($j = 0, 1, 2$) at each node $n$ in the scenario tree. Define $P^*$ to be the probability vector of dimension $1 + N + N^2$ modeled by each $p^*_n$, and $S$ to be the outcome matrix of dimension $3 \times (1 + N + N^2)$ modeled by each $s_{jn}^j$. Therefore, the statistical property estimates are functions of $N$, $S$ and $P$. Let $f_i(N, S, P^*)$ be the mathematical expression for statistical property $i$ in $F$.

Generating a scenario tree that matches the 33 specified statistical properties in $F$ is finally modeled as the following non-convex optimization problem:

$$\min_{(N, S, P^*)} \sum_{i=1}^{33} (f_i(N, S, P^*) - f_{VAL, i})^2$$

s.t. $P^*$ is a probability measure

(6.1)

According to Hoyland and Wallace [11], the optimal value (6.1) is 0 if number $N$ is large enough, we say the scenario tree that leads to $\sum_{i=1}^{33} (f_i(N, S, P^*) - f_{VAL, i})^2 = 0$ is a perfect match of the specified statistical properties $F$. Table 3 shows one of scenario trees calculated by Hoyland and Wallace [11] that perfectly matches the specified statistical properties $F$ in this example.

### 6.2 $\beta$-CVaR pricing

The second step to implement our method is to calculate the no $\rho$-arbitrage interval for specified risk measure $\rho$ based on the scenario tree generated by the Holyand and Wallace’s method[11].

According to Artzner et al.[1], for any convex and closed subset $\mathcal{P}$ of the set of all probability measures set, the risk measure given by $\rho_{\mathcal{P}}(X) = \max_{P \in \mathcal{P}} E^P(X)$ is a coherent risk measure. This means that there exists an infinite number of coherent risk measures. Unfortunately, it is difficult to give a much more meaningful definition of coherent risk measure for most of the convex and closed subsets $\mathcal{P}$.

Three well known coherent risk measures for net worth $X_T$ are:

1) Maximum loss: $\rho(X_T) = \max_{n \in N_T} (-X_n)$, with $\mathcal{P}_\rho = \{ P \mid P$ is a probability measure $\}$;
Table 3: A perfect match scenario tree generated by Hoyland and Wallance [11]'s method

<table>
<thead>
<tr>
<th>Period 1</th>
<th>Period 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_n^*$</td>
<td>$(s_n^1, s_n^2, s_n^3)$</td>
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<tr>
<td>0.005</td>
<td>$(1.057, 74, 138)$</td>
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<tr>
<td>0.085</td>
<td>$(1.046, 78, 205.2)$</td>
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<tr>
<td>0.306</td>
<td>$(1.048, 122.5, 251.8)$</td>
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<tr>
<td>0.123</td>
<td>$(1.062, 93, 172)$</td>
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<tr>
<td>0.059</td>
<td>$(1.034, 114.6, 142)$</td>
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<tr>
<td>0.422</td>
<td>$(1.035, 106.6, 216.8)$</td>
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II) Mean absolute semi-deviation (MASD): 
\[ \rho(X_T) = E^{P^*}(-X_T) + \lambda E^{P^*}(X_T - E^{P^*}(X_T))^+, \]
with \( \lambda \in (0, \frac{1}{2}) \), and \( \mathcal{P}_\rho = \{ Q \mid Q \text{ is a probability measure, and } (\frac{\lambda+1}{2\lambda+1})p^*_n \leq q_n \leq (1 + \lambda)p^*_n \}. \)

III) Conditional value at risk (CVaR) (Rockafellar and Uryasev [20]): 
\[ \rho(X_T) = \min \alpha \left\{ \alpha + \frac{1}{1-\beta} E^{P^*}([-X_T - \alpha]^+) \right\}, \]
where \([x]^+ = \max\{x, 0\}\).

The main results in this thesis hold when the coherent risk satisfies the condition \( P^* \in \text{int} \mathcal{P}_\rho \). Theoretically, this condition does not hold for most of the coherent risk measures, we can see that the assumption \( P^* \in \text{int} \mathcal{P}_\rho \) holds for the three coherent risk measures above.

In this example, we use the conditional value at risk (CVaR) as the risk measure. This measure is defined based on a risk measure known as the Value at Risk (VaR).

For any \( \beta \in (0, 1) \), risk measure \( \beta \)-VaR of loss \(-Z_T^\top \Theta_T\) is defined by
\[ \Phi(Z_T^\top \Theta_T) = \min\{\alpha \mid P^*(-Z_T^\top \Theta_T \leq \alpha) \geq \beta\} \]

The \( \beta \)-CVaR of loss \(-Z_T^\top \Theta_T\) is the mean of the \( \beta \)-tail distribution of \(-Z_T^\top \Theta_T\), and can be equivalently rewritten as 
\[ \rho(Z_T^\top \Theta_T) = \text{mean of the } \beta \text{-tail distribution of } -Z_T^\top \Theta_T, \]

and can be equivalently rewritten as 
\[ \rho(X_T) = \min \alpha \left\{ \alpha + \frac{1}{1-\beta} E^{P^*}([-X_T - \alpha]^+) \right\}. \]

By the research of Rockafellar and Uryasev [21], the \( \beta \)-CVaR of the loss \(-Z_T \Theta_T\) is the mean of \( \beta \)-tail distribution of \(-Z_T \Theta_T\), therefore the set of probability measures associated with the \( \beta \)-CVaR is
\[ \mathcal{P}_\rho = \{ P \mid P \text{ is a probability measure and } 0 \leq p_n \leq \frac{p^*_n}{1-\beta} \text{ for each } n \in \mathcal{N}_T \}, \]
and \( Q = \text{int}(\mathcal{P}_\rho) \cap \mathcal{M} \) is given as
\[ Q = \{ P \mid P \text{ is a martingale probability measure and } 0 < p_n < \frac{p^*_n}{1-\beta} \text{ for each } n \in \mathcal{N}_T \}, \]
Take a single period example with three terminal nodes scenario tree (i.e., $T = 1$, $\mathcal{N}_T = \{1, 2, 3\}$). The sets of $\mathcal{M}$ and $\mathcal{Q}$ are shown as in Figure 7 where $\mathcal{M}$ is the line segment connecting A and B while the set $\mathcal{Q}$ is the set of points between C and D.

Figure 7: A single period, three terminal nodes example of $\mathcal{Q}$ associated with $\beta$-CVaR

When $\beta > 1 - \min_{n \in \mathcal{N}_t} p_n^*$, one has $\mathcal{Q} = \mathcal{M}$, the no $\beta$-CVaR arbitrage interval coincides with the no arbitrage interval that calculated by the fundamental theorem of asset price. The set $\mathcal{Q}$ becomes smaller as $\beta$ decreases from 1 to 0. The interval

$$
\left[ \min_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n Y_n, \max_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n Y_n \right]
$$

becomes smaller as well.

Table 4 shows the no $\rho$ arbitrage interval calculated by our method for $\beta$ decreasing from 1 to 0.4573, where Gap=Max-Min, with Max = $\max_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n Y_n$, and Min = $\min_{Q \in \mathcal{Q}} \frac{1}{\delta_0} \sum_{n \in \mathcal{N}_T} q_n Y_n$. The result of the fundamental theorem of asset pricing shows the no arbitrage interval is (5.52, 18.95). When $\beta = 0.4573$, the no $\beta$-arbitrage price is 10.98, meaning that if the European call option is traded at 10.98, the buyer and seller face to the same risk. In other words, the fair price of this European call option should be 10.98.
Table 4: Results of applying the proposed method when $\beta$-CVaR is employed as the risk measure

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>1</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
<th>0.5</th>
<th>0.4573</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max</td>
<td>18.95</td>
<td>15.7</td>
<td>15.24</td>
<td>14.56</td>
<td>13.36</td>
<td>11.77</td>
<td>10.98</td>
</tr>
<tr>
<td>Gap</td>
<td>12.04</td>
<td>8.78</td>
<td>8.15</td>
<td>6.34</td>
<td>3.82</td>
<td>1.18</td>
<td>0</td>
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</table>

6.3 Comparison with other approaches

There are several other methods for contingent claim pricing in incomplete market when the assets price process follows a scenario tree. Since it is impossible to perfectly hedge a contingent claim in the incomplete market, one of these methods is to generate a portfolio that minimizes the hedge error, the claim’s price is then considered to be the cost of generating this portfolio. If we use the square norm to measure the error, the method can be represented by the problem below:

$$\min_{\Theta^*} \mathbb{E}^P (Z_T^\top \Theta_T - Y_T)^2$$

subject to

$$Z_n^\top [\Theta_n - \Theta_a(n)] = 0 \quad (n \in \mathcal{N}, 1 \leq t \leq T)$$

(6.2)

This method is referred to as the minimum variance hedge (MV) approach. If $\Theta^*$ is an optimal solution of (6.2), then the claim price is given by $Z_0^\top \Theta_0^*$.

If we use the absolute deviation for measuring the error, the method can be represented by the problem below:

$$\min_{\Theta^*} \mathbb{E}^P \left| Z_T^\top \Theta_T - Y_T \right|$$

subject to

$$Z_n^\top [\Theta_n - \Theta_a(n)] = 0 \quad (n \in \mathcal{N}, 1 \leq t \leq T)$$

(6.3)

This method is referred to as absolute the minimum absolute deviation hedge (MAD) approach. If $\Theta^*$ is an optimal solution of (6.3), then the claim price is given by $Z_0^\top \Theta_0^*$.

Table 5 shows the results of applying the MV and MAD approaches. The price of asset computed by MV approach is 11.64, which is a no arbitrage price consistent with
Table 5: Results of applying MV and MAD approaches

<table>
<thead>
<tr>
<th></th>
<th>MV approach</th>
<th>MAD approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>11.64</td>
<td>7.49</td>
</tr>
</tbody>
</table>

the fundamental theorem of asset pricing. However, by the meaning of $\beta$-CVaR arbitrage with $\beta \leq 0.5$, there exists a $\rho$-arbitrage opportunity for the seller if the asset is traded at this price.

On the other hand, by the meaning of the $\beta$-CVaR arbitrage with $\beta \leq 0.8$, there exists arbitrage opportunity for the buyer if the asset is traded at the price 7.49, which is computed by the MAD approach.

7 Conclusion

The fundamental theorem of asset pricing is a theorem of financial engineering that relates the existence of an equivalent martingale measure to the no-arbitrage condition. It has origins in Cox and Ross’s method of risk neutral valuation [5], which was formulated by Harrison and Kreps [9] and Harrison and Pliska [10]. Today, their methodology and the fundamental theorem of asset pricing represent the primary approach that financial engineers use to price contingent claims. It is a standard assumption of economics that markets are arbitrage free. If we make that assumption, the fundamental theorem of asset pricing tells us that there is an equivalent martingale measure, and we can use it to calculate asset prices as expected values.

In a complete market, there exists a unique equivalent martingale probability measure, and we can calculate the exact price of asset. While in an incomplete market, the fundamental theorem of asset pricing (FTAP) shows that the asset must be traded at the price in a specific interval linked to a set of martingale probability measures, but no more information about the price can be provided by the FTAP.

We developed a new and general type of arbitrage, which we referred to as the $\rho$-
arbitrage, by relaxing the no risk constraints.

A $\rho$-arbitrage is a chance of gain through trading without taking much risk. Since it is much more difficult to find traditional arbitrage opportunities in the market than decades ago, a weaker kind of arbitrage, $\rho$-arbitrage will be eliminated by the market force as well. Then it is reasonable to assume that there is no $\rho$-arbitrage in the market. Under this assumption, we provided a no arbitrage interval in the incomplete market that is much narrower than that of the fundamental theorem of asset pricing may provide. Furthermore, we can modify the length of interval by adjusting a parameter of the risk measure $\rho$, such as $\beta$ of $\beta$-CVaR. Finally, our numerical example shows that the model we developed can be used for pricing claim given several statistical properties. Further, it is possible to obtain unique price of asset, which means that the incomplete market could be treated as a complete market by choosing a proper risk measure. When the asset was traded at this price, the buyer and seller faced the same amount of risk.

Although we have proved that the no arbitrage interval of the price of American style claims can be calculated by solving a pair of optimization problems, developing an efficient algorithm to solve these optimization problems still remains to us.

Another future work, which has two directions, is to improve the practicality of the proposed model. The first direction is to extend the model to the continuous stochastic process, and the second one is to develop an efficient algorithm for solving the optimization problems for large scenario trees.
References


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