Non-flippable digraph without making directed 3-cycles

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Abstract

We show the existence of a digraph without directed 3-cycles with the property that every flipping of the direction of one edge produces a directed 3-cycle. This is the first explicitly constructed example of graphs having this property. This construction is only based on a very trivial fact that if a flippable edge is identified with a non-flippable edge, then the edge becomes non-flippable, and after succesive identification, we finally make the flippable edges identified with non-flippable edges. I hope that the reader can understand how easily such graphs can be constructed.

This brief manuscript was originally written to be included into a paper which is no longer completed.

1 Construction of the digraph

(0) Let C be a directed n-cycle $\overrightarrow{c_1 \cdots c_n c_1}$, and v and w two vertices outside of C with edges $\overrightarrow{vc_i}$ and those $\overrightarrow{c_i w}$. We call such a graph a double-wheel. Note that the edges $\overrightarrow{vc_i}$ and $\overrightarrow{c_i w}$ are non-flippable without making directed 3-cycles, while the edges $\overrightarrow{c_i c_{i+1}}$ are flippable.

(1) Let B be a directed cycle of size 4. Let us add 2 double-wheels whose directed cycles C_1 and C_2, respectively, are size 4 as Figure 1. Then the resulting graph $H^{(1)}$ has all the edges non-flippable except the eight edges in the two directed cycles C_1 and C_2. Note that the graph $H^{(1)}$ has no directed 3-cycles.

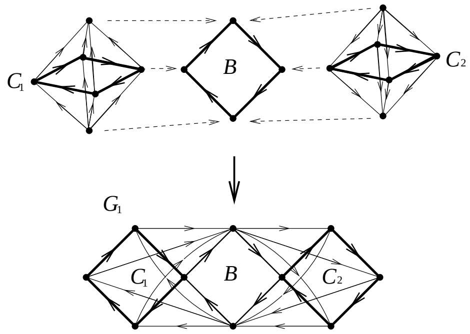

Figure 1: The graph $H^{(1)}$.

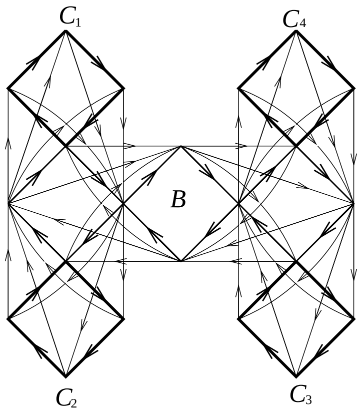

Figure 2: The graph $H^{(2)}$.

(2) As same as step (1), we add a pair of double-wheels to each flippable cycle as Figure 2. The resulting graph $H^{(2)}$ has all the edges non-flippable except 16 edges in 4 directed cycles. Let us call the center cycle B and 4 flippable cycles C_1, C_2, C_3 and C_4. The graph $H^{(2)}$ still has no directed 3-cycles.

(3) Let $H_1^{(2)}$, $H_2^{(2)}$, $H_3^{(2)}$ and $H_4^{(2)}$ be 4 copies of $H^{(2)}$ of step (2). Let us denote the center cycle of H_i by B_i.

Now identify the flippable cycles C_j of 4 graphs $H_i^{(2)}$, for each j. The resulting graph $H^{(3)}$ has 4 flippable cycles C_j $(1 \leq j \leq 4)$ and 4 non-flippable cycles B_k $(1 \leq k \leq 4)$, where the cycles C_j, B_k are mutually disjoint. Other edges are all non-flippable. Observe that this identification does not make no directed 3-cycles.

(4) Let $H_1^{(3)}$, $H_2^{(3)}$, $H_3^{(3)}$ and $H_4^{(3)}$ be n (≥ 4) copies of $H^{(3)}$ of step (3), and each graph $H_k^{(3)}$ have 4 flippable cycles $C_i(H_k^{(3)})$ $(1 \leq i \leq 4)$ and non-flippable cycle $B_i(H_k^{(3)})$ which are mutually disjoint.

Now identify the cycle $C_i(H_k^{(3)})$ with $B_i(H_{k+1}^{(3)})$ for each fixed pair (i, j, k), where the summation is taken in modulo n. The resulting graph G has no directed 3-cycles and no flippable edges.