Orientations on simplicial complexes and cubical complexes

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Abstract

In this paper we discuss orientations of the facet-ridge incidence graphs of regular CW complexes, especially the cases of simplicial complexes and cubical complexes. When the orientation is acyclic and each ridge has in-degree ≥ 1, it gives a covering by a family of sets associated to each facets. We discuss the case when the covering becomes a partition. For simplicial complexes it provides a characterization of shellability, and for cubical complexes we derive inequalities for the Betti numbers of the complexes. Also some additional examples are supplied.

1 Acyclic orientations of regular CW complexes and simplicial complexes

When we treat cell complexes such as simplicial complexes or polytopal complexes in a combinatorial context, in many cases the most general interesting class is that of regular CW complexes. See [2, 3] for the definition and other details.

Let C be a regular CW complex. The maximal faces with respect to the inclusion relation are *facets*, and we define *ridges* as the faces which are covered only by facets. (We say a face σ covers τ when $\sigma \supset \tau$ and $\dim \sigma = \dim \tau + 1$.) The faces that are not ridges but covered by facets are *pseudoridges*. Note that the empty set is considered as a member of every regular CW complex. When we treat CW complexes without the empty set, we use the notation $\check{C} = C \setminus \{\emptyset\}$.

Let $G(C)$ be the facet-ridge incidence graph, i.e, the graph whose nodes are facets and ridges of C, and a facet σ and a ridge τ are connected by an edge if τ is a face of σ. Also we let $G'(C)$ be the graph to which the incidences between facets and pseudoridges are appended. Throughout this paper, we discuss orientations of $G(C)$. For a given orientation O to $G(C)$, we extend the orientation for $G'(C)$ by additionally giving orientations from each pseudoridge to its covering facets, and from other facets to the pseudoridge. We write $G^O(C)$ and $G'^O(C)$ the directed graphs derived from $G(C)$ and $G'(C)$ oriented by O. See Figure 1. (In the figure, black nodes are facets, white nodes are ridges, and doubled nodes are pseudoridges.)

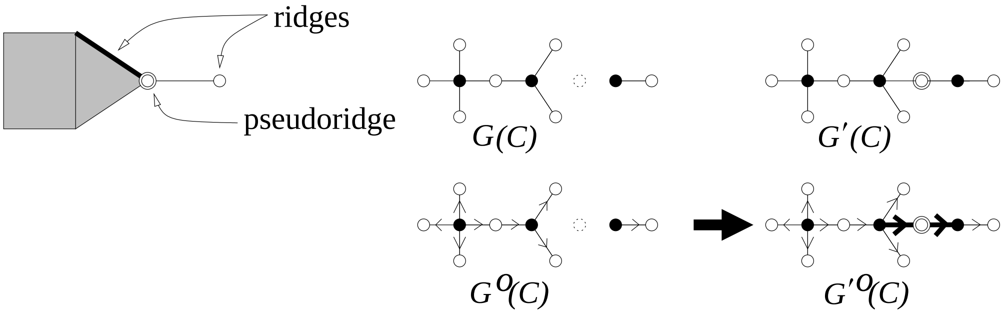


Figure 1: $G(C)$, $G'(C)$ and their orientations $G^O(C)$ and $G'^O(C)$.

We define the set $S^O(\sigma)$ for each facet σ as follows.

$$S^O(\sigma) = \{\eta \in C : \sigma \to \tau \text{ in } G'^O(C) \text{ for every (pseudo)ridge } \tau \text{ with } \eta \subseteq \tau \subseteq \sigma\} \cup \{\sigma\}.$$

Note that the set of faces of σ which is not belonging to $S^O(\sigma)$ equals to

$$\begin{aligned} S^{cO}(\sigma) &= \{\eta \in C : \tau \to \sigma \text{ in } G'^O(C) \text{ for some (pseudo)ridge } \tau \text{ with } \eta \subseteq \tau \subseteq \sigma\ \} \\ &= \bigcup \{\overline{\tau} : \tau \to \sigma \text{ in } G'^O(C)\ \}, \end{aligned}$$

where $\overline{\tau}$ is the set of all faces of τ. We use the notation $\check{S}^O(\sigma) = S^O(\sigma) \setminus \{\emptyset\}$.

By letting $G'^O_{\supseteq\eta}(C)$ be the subgraph of $G'^O(C)$ induced by the facets, ridges and pseudoridges α with $\alpha \supseteq \eta$, we have the following lemma which can be easily verified.

Lemma 1. *For each $\eta \in C$, $\eta \in S^O(\sigma)$ if and only if σ is a source in $G'^O_{\supseteq\eta}(C)$.* □

The orientation O is *acyclic* if it has no directed cycles, and *admissible* if the in-degree of each ridge is at least 1. If O is acyclic, then, for each $\eta \in C$, $G'^O_{\supseteq\eta}(C)$ is acyclic thus has at least one source node. Further, if O is admissible, then the source nodes must be facets. Thus we have the following proposition.

Proposition 2. *If O is acyclic and admissible, then $\bigcup\{S^O(\sigma) : \sigma \text{ is a facet of } C\} = C$. Thus we have*

$$\min_{\substack{O \text{ is acyclic} \\ \text{and admissible}}} \sum_{\sigma:\text{ facet of } C} |S^O(\sigma)| \ \geq\ f(C),$$

where $f(C)$ is the number of faces of C. The equality holds if and only if there is an acyclic and admissible orientation O for which $\{S^O(\sigma)\}$ gives a partition of C. □

We define a directed graph $\widetilde{G}^O(C)$ whose nodes are facets and arcs $\sigma \to \sigma'$ are defined if there is a face η with $\eta \subset \sigma'$ and $\eta \in S^O(\sigma)$. We have the following lemma.

Lemma 3. *When $\{S^O(\sigma)\}$ is a partition of C, $\widetilde{G}^O(C)$ is acyclic if and only if $G^O(C)$ is acyclic.*

Proof. Note that $G^O(C)$ is acyclic if and only if $G'^O(C)$ is acyclic.

Assume $G'^O(C)$ is acyclic. Let $\sigma \to \sigma'$ is an arc of $\widetilde{G}^O(C)$. From the definition of $\widetilde{G}^O(C)$, there is a face η with $\eta \subset \sigma'$ and $\eta \in S^O(\sigma)$. Here, both σ and σ' are nodes of $G'^O_{\supseteq\eta}(C)$, and, from Lemma 1 and the assumption that $\{S^O(\sigma)\}$ is a partition, σ is a unique source of $G'^O_{\supseteq\eta}(C)$. Since $G'^O_{\supseteq\eta}(C)$ is acyclic, this implies that there should be a directd path from σ to σ' in $G'^O_{\supseteq\eta}(C)$, thus in $G'^O(C)$. We conclude that $\widetilde{G}^O(C)$ can have no directed cycles.

Assume $G^O(C)$ has a directed cycle. The cycle should be of the type $\sigma_1 \to \tau_1 \to \sigma_2 \to \tau_2 \to \cdots \to \sigma_s \to \tau_s \to \sigma_1$, where σ_i's are facets and τ_i's are ridges. Then for each i, we have $\tau_i \subset \sigma_i$ and $\tau_i \in S^O(\sigma_{i+1})$. By the definition of $\widetilde{G}^O(C)$, this implies that there is a directed cycle $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_s \to \sigma_1$ in $\widetilde{G}^O(C)$. □

Assume $\{S^O(\sigma)\}$ is a partition for an acyclic admissible orientation O. Then Lemma 3 assures that $\widetilde{G}(C)$ is acyclic. Let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be a linear extension of $\widetilde{G}^O(C)$. Then we have the following proposition.

Proposition 4. *Under the assumption above, we have the following.*

- $C^{(i)} = \bigcup_{j=1}^{i} S^O(\sigma_j)$ *forms a regular CW complex (*$= \bigcup_{j=1}^{i} \overline{\sigma_j}$*) for each* $1 \le i \le t$.
- $C^{(i-1)} \cap \overline{\sigma_i} = S^{cO}(\sigma_i)$ *for each* $2 \le i \le t$.

□

The theory from Lemma 1 to Proposition 4 works completely in parallel if we regard the empty set not as a member of CW complexes by replacing C by $\check{C}$. (Accordingly, suitably replace $S^O(\sigma)$ by $\check{S}^O(\sigma)$, $S^{cO}(\sigma)$ by $\check{S}^{cO}(\sigma)$, $f(C)$ by $f(\check{C})$). We use this version in Section 3.

From these lemmas and propositions in the general setting of regular CW complexes, we can directly derive the following theorem for the case of simplicial complexes. For the definition and details of shellability, see [2, 3, 11], for example.

Theorem 5. *Let* C *be a simplicial complex. Then*

$$\min_{\substack{O \text{ is acyclic} \\ \text{and admissible}}} \sum_{\sigma:\text{ facet of } C} 2^{\deg_{\mathrm{out}}(\sigma)} \ \ge\ f(C),$$

and the equality holds if and only if C *is shellable, where* $f(C)$ *is the number of faces of* C*, and* $\deg_{\mathrm{out}}(\sigma)$ *is the out-degree of* σ *in* $G^O(C)$.

Proof. For simplicial complexes, $|S^O(\sigma)| = 2^{\deg_{\mathrm{out}}(\sigma)}$. Thus the inequality follows from Proposition 2, and the equality holds if and only if $\{S^O(\sigma)\}$ is a partition, where O is an orientation that achieves the minimum. The only if part is as follows. From Lemma 3, $\tilde{G}(C)$ is acyclic. Let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be a linear extension of $\tilde{G}(C)$. Proposition 4 shows that the linear extension is a shelling since $(\bigcup_{j=1}^{i-1} \sigma_j) \cap \sigma_i = S^{oC}(\sigma_i)$ is generated only by the ridges of σ_i. For the if part, let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be a shelling of C. Let an orientation O be such that $\tau \to \sigma_i$ if $\tau \subset \sigma_j$ for some $j < i$, and $\sigma_i \to \tau$ otherwise, for each ridge τ covered by σ_i. Then $\{S^O(\sigma_i)\} = \{\eta \subseteq \sigma_i : \eta \not\subset \sigma_k \text{ with } k < i\}$, which shows that $\{S^O(\sigma_i)\}$ becomes a partition. □

Theorem 5 appeared in [7]. The proof is slightly different but in the same line. The construction of the if part appears essentially in [9].

Remark. In [9], an assignment $\chi(\tau,\sigma) \in \{+,-\}$ for each pair of (pseudo)ridge τ and its covering facet σ is called a facing. Giving an orientation O to $G'(C)$ in our setting can be seen as giving a facing to C by translating $\chi(\tau,\sigma) = +$ by $\sigma \to \tau$ in $G'^O(C)$ and $\chi(\tau,\sigma) = -$ by $\tau \to \sigma$ in $G'^O(C)$. Under this translation and the terminology of [9], the facing χ is exact if and only if $G'^O_{\supseteq\eta}(C)$ has exactly one facet as its source for each $\eta \in C$. Thus, from Lemma 1, there is an (not necessarily acyclic) orientation O of $G'(C)$ for which $\{S^O(\sigma)\}$ gives a partition if and only if C is signable. (The definition of signability in [9] is given only for pure cases, but it can be naturally generalized for nonpure cases.) Thus Proposition 2 can be seen as giving a sufficient condition of C to be signable. But the condition is stronger than signability because the partition in Proposition 2 has additional condition that $\widetilde{G}^O(C)$ to be acyclic from Lemma 3. This difference can be observed in Theorem 5. A simplicial complex is signable if and only if it is partitionable, but Theorem 5 characterizes shellability, a stronger property than partitionability. It seems difficult to give an optimization-type characterization of signability or partitionability similar to Proposition 2 or Theorem 5. We also remark that Theorem 5 for simplicial polytopes essentially given in [8].

2 Some examples of simplicial complexes

In Figure 2, for each pair, (a) and (b) have isomorphic facet-ridge incidence graphs, but (a) is shellable while (b) is nonshellable. There are many other realizations for these incidence graphs. (In the figure, the black nodes corresponds to facets, and white nodes corresponds to ridges.) There are examples even for triangulated closed manifolds, See Figure 3 These examples shows that shellability can not be characterized only from the facet-ridge incidence graphs. But Theorem 5 shows that we can tell whether the simplicial complex is shellable or not by additionally counting the number of faces.

In the converse way, let us consider the following problem.

> Given a bipartite graph $G = (V_1, V_2, E)$, determine the maximum number of faces of a simplicial complex whose facet-ridge incidence graph is isomorphic to G with $V_1 = \{\text{facets}\}$ and $V_2 = \{\text{ridges}\}$.

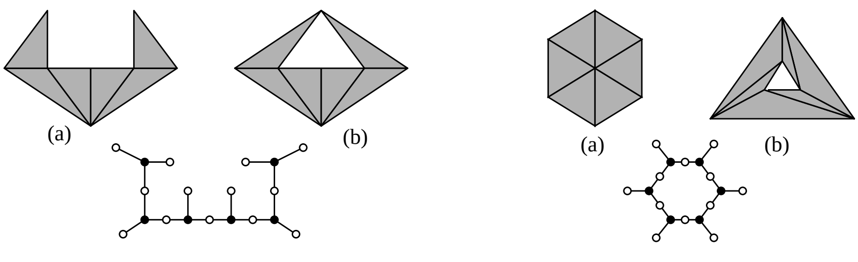


Figure 2: Shellable and nonshellable realizations

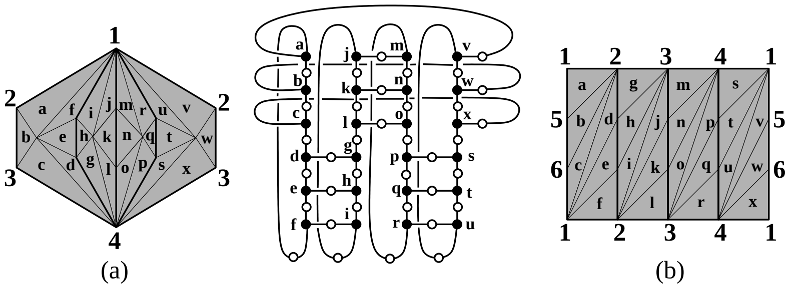


Figure 3: Example of triangulated closed manifold

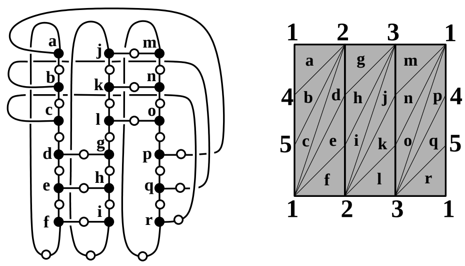


Figure 4: An example with no shellable realizations

From Theorem 5, if we somehow construct a shellable simplicial complex whose facet-ridge incidence graph is G, then we can tell that it achieves the maximum. For example, for the graph in the left of Figure 2, 28 is the maximum number of faces.

But shellable realizations do not always exist. The graph in Figure 4 has no shellable realization. (The simplicial complex given in the figure is an example of nonshellable realization.)

3 Acyclic orientations on cubic complexes

Proposition 2 does not characterize shellability of regular CW complexes in general. For example, in the cubic complex of Figure 5, the orientation achieves the equality of Proposition 2 but it is not shellable. (The figure in the right is the partition $\{S^O(\sigma)\}$ induced by the orientation.)

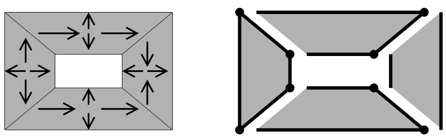


Figure 5: Nonshellable cubic complex

For each facet σ of a cubical complex C, σ covers $\dim\sigma$ antipodal pairs of (pseudo)ridges. For an orientation O of $G'(C)$, we define $(t_0(\sigma), t_1(\sigma), t_2(\sigma))$ the type of the facet σ, where

- $t_0(\sigma)$ is the number of antipodal pairs of (pseudo)ridges $\{\tau, \tau'\}$ with $\sigma \to \tau$ and $\sigma \to \tau'$,
- $t_2(\sigma)$ is the number of antipodal pairs of (pseudo)ridges $\{\tau, \tau'\}$ with $\sigma \leftarrow \tau$ and $\sigma \leftarrow \tau'$,
- $t_1(\sigma)$ is the number of the rest of antipodal pairs.

The same definition can be found in [1]. Further, we define $p_i(C) = \#\{\sigma : t_1(\sigma) = 0 \text{ and } t_2(\sigma) = i\}$.

Proposition 2 and Proposition 4 imply the following theorem. (We use the version of $\check{C}$ as remarked after Proposition 4.)

Theorem 6. *Let C be a cubical complex. Then*

$$\min_{\substack{O \text{ is acyclic} \\ \text{and admissible}}} \sum_{\sigma:\ \text{facet of } C} 2^{t_1(\sigma)} 3^{t_0(\sigma)} \;\geq\; f(\check{C}). \tag{1}$$

When the equality holds, we have the inequalities

$$\beta_k(C) - \beta_{k-1}(C) + \cdots + (-1)^{k-1}\beta_0(C) \leq p_k(C) - p_{k-1}(C) + \cdots + (-1)^{k-1}p_0(C) \qquad (0 \leq k \leq \dim C)$$

$$p_0(C) - p_1(C) + \cdots + (-1)^{\dim C - 1} p_{\dim C}(C) = \chi(C),$$

$$\beta_i \leq p_i, \quad (0 \leq i \leq \dim C)$$

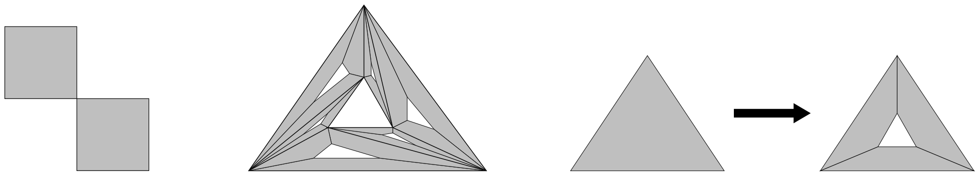
where $\beta_i(C)$ is the i-th Betti number of C, and $\chi(C)$ is the Euler characteristic of C.

Proof. When σ is a facet of type (t_0, t_1, t_2) under the orientation O, then $|\check{S}^O(\sigma)| = 2^{t_1(\sigma)}3^{t_0(\sigma)}$. Thus the inequality follows from Proposition 2. The equality holds if and only if $\{\check{S}^O(\sigma)\}$ is a partition, where O is an orientation that achieves the minimum. In this case, from Lemma 3, $\widetilde{G}(C)$ is acyclic. Let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be a linear extension of $\widetilde{G}(C)$. Proposition 4 shows that $C^{(i)} = \bigcup_{j=1}^{i} S^O(\sigma_j)$ forms a regular CW complex (cubical complex) for each $1 \leq i \leq t$. Here we have the following observation.

- If $t_1(\sigma_i) \geq 1$, $S^{cO}(\sigma_j)$ is shellable (see, for example, [11, Exercise 8.1(i)]), thus $C^{(i)}$ is homotopy equivalent to $C^{(i-1)}$.
- If $t_1(\sigma_i) = 0$, then $C^{(i)}$ is homeomorphic to $C^{(i-1)} \cup_{(I^{t_0(\sigma_i)} \times \{0,1\}^{t_2(\sigma_i)})} (I^{t_0(\sigma_i)} \times I^{t_2(\sigma_i)})$, where I is a unit interval. Thus $C^{(i)}$ is homotopy equivalent to $C^{(i-1)} \cup_{\partial B^{t_2(\sigma_i)}} B^{t_2(\sigma_i)}$, where $B^{t_2(\sigma_i)}$ is a $t_2(\sigma_i)$-dimensional cell.

From these observations we conclude that C is homotopy equivalent to a CW complex with p_i i-cells for each i. The inequalities follow from this fact according to the standard arguement in Morse theory, see for example [6, 10]. $\square$

In order to get a characterization of shellability, we need additional condition for the types of facets. The proof is immediate from the proof of Theorem 6.



(A) (B)

Figure 6: An example the equality can not be achieved.

Theorem 7. *Let the orientation O achieve the equality of (1). If $\#\{\sigma : 0 < t_2(\sigma) < \dim \sigma,$ and $t_1(\sigma) = 0\} = 0$ and $\#\{\sigma : t_0(\sigma) = \dim \sigma\} = 1$, then C is shellable.* □

Theorem 7 is especially useful when the cubic complex C is a cubical decomposition of a manifold. The following corollary is an analogy of the sphere theorem. (See [6].)

Corollary 8. *Let C be a cubical decomposition of a closed manifold. If C has an orientation O that achieves the equality of (1) with $p_0 = p_d = 1$ and $p_i = 0$ for $0 < i < d$, then C is a PL sphere.*

Proof. By Theorem 7, C is shellable. It is well-known that if a regular CW decomposition of a closed manifold is shellable, then it is a PL sphere. See, for example, [3]. □

Remark. As same as the simplicial cases, there are cubical complexes such that no orientation achieves the equality of (1) in Theorem 6. For example, Figure 6 ((a) and (b)) shows such examples. Note that the example (B) is derived from a two-dimensional simplicial complex with 6 facets by replacing each facet with three cubic facets as shown in the right of the figure. To show that the equality can not be achieved for this example, observe that a two-dimensional simplicial complex is shellable if and only if the cubical complex derived in this way has no orientation achieving the equality. (The same construction also works for higher dimensions.)

If there is no orientation achieving the equality of (1), then Theorem 6 is of no use. But, if every link of a cubical complex C is shellable (for example, Figure 6 (b) satisfies this condition), the cubical complex C' derived by replacing every facet σ by $\overbrace{3 \times 3 \times \cdots \times 3}^{\dim \sigma}$ cubes as shown in Figure 7 has always orientations achieving the equality.
(For this, first observe that there is a natural correspondence between facets of $\check{C}'$ and an incidence (η, σ) between a face η and a facet σ of $\check{C}$. A construction is as follows. List the faces of $\check{C}$ in the ordering that faces of small dimensions come first. Then for each face η of $\check{C}$ in the list, replace the face by a shelling of $\mathrm{link}_C(\eta)$. If we replace the facet ξ appeared in the shelling of $\mathrm{link}_C(\eta)$ by the facet of C' correspondent to the incidence $(\eta, \eta * \xi)$, we get an ordering of facets $\sigma_1, \sigma_2, \ldots, \sigma_t$ of $\check{C}'$. By setting an orientation O of $G(C')$ from this ordering such that $\tau \to \sigma_i$ if $\tau \subset \sigma_j$ for some $j < i$, and $\sigma_i \to \tau$ otherwise, for each ridge τ covered by σ_i, it is easy to verify that O is acyclic and admissible, and $\{\check{S}^O(\sigma_i)\}$ becomes a partition of $\check{C}'$.)

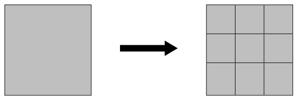


Figure 7: Replacing a cube by $3 \times 3 \times \cdots \times 3$ cubes.

Because C and C' have the same Betti numbers, using C' instead of C may give some information from Theorem 6, though using C' may provide large p_i's.

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