Hereditary-shellable simplicial complexes

Masahiro HACHIMORI
Faculty of Engineering, Information and Systems,
University of Tsukuba
Tsukuba, Ibaraki 305-8573, Japan
hachi@sk.tsukuba.ac.jp

Kenji KASHIWABARA
Department of General Systems Studies,
University of Tokyo
Komaba, Meguro, Tokyo 153-8902, Japan
kashiwa@idea.c.u-tokyo.ac.jp

Jun, 2014

Abstract

Hereditary-shellable simplicial complexes are those such that any restrictions, including themselves, are shellable. Matroid complexes are examples of hereditary-shellable simplicial complexes. We show that the pure 2-skeletons of 2-dimensional hereditary-shellable simplicial complexes are extendably shellable. This generalizes the result of Björner and Eriksson that matroid complexes of dimension 2 are extendably shellable. On the other hand, we give an example of a 2-dimensional hereditary-shellable simplicial complex that is not vertex decomposable. Further, we discuss properties of pure skeletons of strong obstructions to shellability as an application of our result.

1 Introduction

In this paper, we investigate simplicial complexes such that any restrictions, including themselves, are shellable. We say such simplicial complexes are hereditary-shellable ([9]). A related concept is "obstructions to shellability," studied by Wachs [13]: an obstruction to shellability is a simplicial complex that is not shellable but all of its proper restrictions are shellable. A trivial but important relation between these two concepts is that a simplicial complex is hereditary-shellable if and only if it has no obstructions to shellability as its restrictions. In other words, obstructions to shellability are the forbidden subcomplexes (with respect to restrictions) for hereditary-shellability.

We have the following implications for general pure simplicial complexes.

$$\text{uniform matroid} \Rightarrow \text{matroid} \begin{array}{c}\not\nearrow\!\!\!\!\nearrow \\ \searrow\!\!\!\!\searrow\end{array} \begin{array}{c}\text{hereditary-shellable}\\ \text{vertex decomposable}\end{array} \begin{array}{c}\searrow\!\!\!\!\searrow\\ \nearrow\!\!\!\!\nearrow\end{array} \text{shellable}$$

Here, the implication "matroid $\Rightarrow$ hereditary-shellable" follows from the fact that any restriction of a matroid complex is again a matroid complex, and that all matroid complexes are shellable (see [2, 10]).

A (pure) shellable complex is extendably shellable if every partial shelling can be extended to the whole shelling ([8]). Björner and Eriksson [3] showed that 2-dimensional matroid complexes are extendably shellable. In this paper we show that the pure 2-skeletons of a 2-dimensional hereditary-shellable complexes are extendably shellable (Theorem 3.5). Especially, pure 2-dimensional hereditary-shellable simplicial complexes are extendably shellable (Corollary 3.6). This strengthens the result of Björner and Eriksson, and give the following refined implications for pure 2-dimensional simplicial complexes.

$$\text{uniform matroid} \Rightarrow \text{matroid} \begin{array}{c}\nearrow\\ \searrow\end{array} \begin{array}{c}\text{hereditary-shellable} \Rightarrow \text{extendably shellable}\\ \text{vertex decomposable}\end{array} \begin{array}{c}\searrow\\ \nearrow\end{array} \text{shellable}$$

It is natural to ask whether the implication "matroid ⇒ vertex decomposable" is refined similarly or not, that is, whether "hereditary-shellable ⇒ vertex decomposable" holds or not (for dimension 2). Unfortunately, the answer is negative. We show there exist pure hereditary-shellable simplicial complexes that are not vertex decomposable in each dimension ≥ 2 (Example 4.6).

It is currently an open question whether pure 3-dimensional hereditary-shellable complexes are extendably shellable or not. We even do not know whether 3-dimensional matroid complexes are extendably shellable or not. But it should be remarked that Tracy Hall [11] gives an example of an 11-dimensional matroid complex that is not extendably shellable. (This example is the boundary complex of a 12-dimensional crosspolytope.) Therefore, hereditary-shellability does not imply extendable shellability for higher dimensions in general.

In the last section, as an application of our result, we discuss the properties of pure skeletons of 3-dimensional strong obstructions to shellability. Here, a strong obstruction to shellability is a nonshellable simplicial complex such that whose proper restrictions and proper links are all shellable. We show that, if Γ is a 3-dimensional strong obstructions to shellability, the pure 3-skeleton of Γ is nonshellable while the pure i-skeletons of Γ are shellable for $0 \leq i \leq 2$.

2 Preliminaries

Let V be a finite set. $\Gamma \subseteq 2^V$ is a (finite) *simplicial complex* on V if $B \subseteq A \in \Gamma$ implies $B \in \Gamma$. The set V is the set of *vertices* of Γ. A *face* is an element of Γ, and a *facet* is a maximal face with respect to inclusion. For a set $\{F_1, F_2, \ldots, F_n\}$ of faces, we denote the simplicial complex $\{A \subseteq V | A \subseteq F_i \text{ for } 1 \leq {}^\exists i \leq n\}$ by $\langle F_1, \ldots, F_n \rangle$. The *dimension* of a face $A \in \Gamma$ is $|A| - 1$, and the dimension of a simplicial complex is defined as the maximum dimension of its faces. (Note that the empty set is a face of dimension -1.) A simplicial complex is *pure* if all the facets have the same dimension. We use the following conventional terminologies for the names of faces. A face of dimension 0 (= of size 1) is a *vertex*, and a face of dimension 1 (= of size 2) is an *edge*. (We use the same terminology "vertex" for both an element of the underlying set V and a 1-dimensional face.) A face of dimension 2 is a *2-face* or a *triangle*. A face of dimension k is a *k-face*.

Shellability of simplicial complexes is defined as follows.

Definition 2.1. ([4]) A simplicial complex Γ is *shellable* if all the facets can be arranged into a sequence $F_1, \ldots, F_n$ that satisfies the following condition: for each $2 \leq k \leq n$, $\langle F_1, \ldots, F_{k-1} \rangle \cap \langle F_k \rangle$ is a pure simplicial complex of dimension $\dim F_k - 1$.

A pure d-dimensional simplicial complex is *strongly connected* if for any two facets F and G there is a sequence of facets $F = F_1, F_2, \ldots, F_k = G$ such that $|F_i \cap F_{i+1}| = d$ for all $1 \leq i \leq k-1$. The following property is well-known and easily observed.

Lemma 2.2. *(e.g. [2, Ex. 7.3]) A pure shellable simplicial complex is strongly connected.*

Especially, in a pure shellable d-dimensional simplicial complex with $d \geq 1$, the graph consisting of the vertices and the edges of the simplicial complex is always connected.

For a simplicial complex Γ on V, the *link* of $A \in \Gamma$ is defined as $\mathrm{link}_\Gamma(A) = \{F \in \Gamma : F \cup A \in \Gamma, F \cap A = \emptyset\}$. For a simplicial complex Γ and $0 \leq k \leq \dim \Gamma$, the *pure k-skeleton* $\mathrm{pure}_k(\Gamma)$ is the simplicial complex consisting of the k-faces of Γ and its subfaces. Note that $\mathrm{pure}_k(\Gamma)$ is a pure k-dimensional simplicial complex. It is known that these two operations preserve shellability as follows.

Lemma 2.3. *([4, Proposition 10.14]) If a simplicial complex* Γ *is shellable, then* $\mathrm{link}_\Gamma(A)$ *is shellable for any* $A \in \Gamma$.

Lemma 2.4. *([4, Theorem 2.9]) If a simplicial complex* Γ *is shellable, then* $\mathrm{pure}_k(\Gamma)$ *is shellable for all* $0 \leq k \leq \dim \Gamma$.

Note that the converse of Lemma 2.4 is not true for dim ≥ 3 as remarked in [9, p. 1610], but is true for dimensions up to 2: it is trivial for dimensions ≤ 1, and the 2-dimensional case follows from the following lemma. (Remark that $\mathrm{pure}_0(\Gamma)$ is always shellable, and $pure_1(\Gamma)$ is shellable if and only if it is connected.)

Lemma 2.5. *([9, Prop. 2.7]) A 2-dimensional simplicial complex* Γ *is shellable if and only if* $\mathrm{pure}_2(\Gamma)$ *is shellable and* $\mathrm{pure}_1(\Gamma)$ *is connected.*

For a simplicial complex Γ on V, the *restriction* to $A \subseteq V$ is defined as $\Gamma[A] = \{F \in \Gamma : F \subseteq A\}$. We sometimes refer $\Gamma[V - B]$ as the *deletion* of B. Note that restrictions of a pure simplicial complex may not be pure.

In general, restrictions of a shellable simplicial complex may not be shellable. Thus the following definition is a strengthening of shellability.

Definition 2.6. ([9]) A simplicial complex Γ is *hereditary-shellable* if all its restrictions are shellable.

Remark that every hereditary-shellable simplicial complex itself is shellable by considering the restriction to the whole vertex set V. Hereditary-shellable complexes are discussed in relation to the obstructions to shellability in [9].

Example 2.7. $\Gamma_1 = \langle\{a, b, c\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}\rangle$ is shellable but not hereditary-shellable, since $\Gamma_1[\{a, b, e, f\}] = \langle\{a, b\}, \{e, f\}\rangle$ is not shellable. On the other hand, $\Gamma_2 = \langle\{a, b, c\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}, \{a, d\}, \{a, e\}, \{a, f\}, \{b, e\}, \{b, f\}, \{c, f\}\rangle$ is hereditary-shellable.

Example 2.8. The simplicial complex $\Gamma_3 = \Gamma_2 \cup \langle\{\{a, g\}, \{c, g\}, \{e, g\}, \{b, h\}, \{d, h\}, \{f, h\}, \{g, h\}\}\rangle$ is also hereditary-shellable. (This example will be used later in Example 4.6.)

Other important examples are matroid complexes, as will be discussed in Section 4.

We list basic properties about hereditary-shellability as follows.

Lemma 2.9. *Every restriction of a hereditary-shellable simplicial complex is hereditary-shellable.*

Proof. This follows from the definition. ■

Lemma 2.10. *Every link of a hereditary-shellable simplicial complex is hereditary-shellable.*

Proof. This follows from Lemma 2.3 and the relation $(\mathrm{link}_\Gamma(A))[W] = \mathrm{link}_{\Gamma[W\cup A]}(A)$ for $A \in \Gamma$ and $W \subseteq V(\Gamma)$ with $A \cap W = \emptyset$. ■

Lemma 2.11. *Let Γ be a d-dimensional hereditary-shellable simplicial complex, and assume there are facets $A, B \in \Gamma$ with $\dim A = \dim B$ and $|A\triangle B| \geq 4$. Then there exists a facet $C \in \Gamma$, different from A and B, such that $\dim C \geq \dim A$ and $A \cap B \subseteq C \subseteq A \cup B$.*

Proof. Assume there exists no facet C in Γ, different from A and B, satisfying $\dim C \geq \dim A$ and $A \cap B \subseteq C \subseteq A \cup B$. This means there exist no facets of dimension $\geq k = \dim(A - B) = \dim(B - A)$ in $\mathrm{link}_{\Gamma[A\cup B]}(A \cap B)$ other than $A - B$ and $B - A$. That is, $\mathrm{link}_{\Gamma[A\cup B]}(A \cap B)$ has exactly two k-dimensional facets $A-B$ and $B-A$, and other facets are of smaller dimensions. Here, $k = |A-B|-1 = |A\Delta B|/2 - 1 \geq 1$. Since $A - B$ and $B - A$ are disjoint, $\mathrm{pure}_k(\mathrm{link}_{\Gamma[A\cup B]}(A \cap B))$ is not strongly connected. This implies $\mathrm{pure}_k(\mathrm{link}_{\Gamma[A\cup B]}(A \cap B))$ is nonshellable by Lemma 2.2, $\mathrm{link}_{\Gamma[A\cup B]}(A \cap B)$ is nonshellable by Lemma 2.4, and then $\Gamma[A \cup B]$ is nonshellable by Lemma 2.3. But this contradicts the assumption that Γ is hereditary-shellable. ■

Corollary 2.12. *Let Γ be a 2-dimensional hereditary-shellable simplicial complex, and a, b, c, d, e, f be distinct vertices of Γ.*

- *(i) If $\{a, b, c\}$ and $\{c, d, e\}$ are facets of Γ, then one of $\{a, c, d\}$, $\{a, c, e\}$, $\{b, c, d\}$ and $\{b, c, e\}$ is a facet.*
- *(ii) If $\{a, b, c\}$ and $\{d, e, f\}$ are facets of Γ, then there exists a facet $A \in \Gamma$ such that $|A\cap\{a, b, c\}| = 2$ and $|A \cap \{d, e, f\}| = 1$.*

Proof. (i) follows directly from Lemma 2.11. For (ii), Lemma 2.11 assures the existence of a facet $C \in \Gamma$ such that (a): $|C \cap \{a, b, c\}| = 2$ and $|C \cap \{d, e, f\}| = 1$, or (b): $|C \cap \{a, b, c\}| = 1$ and $|C \cap \{d, e, f\}| = 2$. But, it is easily shown (b) implies (a) by applying (i) to $\{a, b, c\}$ and C. ■

For a simplicial complex, consider a sequence $F_1, \dots, F_m$ of facets that does not necessarily consist of all the facets but satisfies the condition in Definition 2.1 for each i with $2 \leq i \leq m \leq n$. Such a sequence is called a *partial shelling*. When a partial shelling uses all the facets, then it is a *complete shelling*. The following definition is proposed by Danaraj and Klee in [8].

Definition 2.13. ([8]) A simplicial complex Γ is *extendably shellable* if any partial shelling $F_1, \dots, F_m$ can be extended to a complete shelling $F_1, \dots, F_m, F_{m+1}, \dots, F_n$.

Note that extendably shellable complexes have to be pure, since every shelling should start from a facet of maximum dimension while any one facet constitutes a partial shelling. It is known that extendable shellability is strictly stronger than shellability even for pure simplicial complexes, see [3], etc.

We say a partial shelling $F_1, \ldots, F_m$ is a *maximal partial shelling* when there exists no partial shelling $F_1, \ldots, F_m, F_{m+1}$. Note that a simplicial complex has at least one maximal partial shelling that is not complete if it is not extendably shellable.

As remarked before, the converse of Lemma 2.4 is not true for dimensions ≥ 3. But, by adding extendability, we have the following. This proposition will play a role in Section 5.

Proposition 2.14. *For a d-dimensional simplicial complex Γ, if $\mathrm{pure}_d(\Gamma)$ is shellable and $\mathrm{pure}_k(\Gamma)$ is extendably shellable for all $0 \leq k \leq d-1$, then Γ is shellable.*

Proof. Construct a shelling of Γ as follows. First, let $\mathcal{S}_d = F_1^d, F_2^d, \ldots, F_{k_d}^d$ be a shelling of $\mathrm{pure}_d(\Gamma)$. Since $\mathrm{pure}_d(\Gamma)$ is shellable, $\mathrm{pure}_{d-1}(\mathrm{pure}_d(\Gamma))$ is shellable by Lemma 2.4. Take a shelling $\mathcal{T}^{d-1} = G_1^{d-1}, G_2^{d-1}, \ldots, G_{l_{d-1}}^{d-1}$ of $\mathrm{pure}_{d-1}(\mathrm{pure}_d(\Gamma))$. This is a partial shelling of $\mathrm{pure}_{d-1}(\Gamma)$. By the assumption that $\mathrm{pure}_{d-1}(\Gamma)$ is extendably shellable, this partial shelling can be extended to a shelling $\mathcal{T}^{d-1}\mathcal{S}^{d-1} = G_1^{d-1}, G_2^{d-1}, \ldots, G_{l_{d-1}}^{d-1}, F_1^{d-1}, F_2^{d-1}, \ldots, F_{k_{d-1}}^{d-1}$ of $\mathrm{pure}_{d-1}(\Gamma)$. From this it is easy to verify that $\mathcal{S}^d\mathcal{S}^{d-1} = F_1^d, F_2^d, \ldots, F_{k_d}^d, F_1^{d-1}, F_2^{d-1}, \ldots, F_{k_{d-1}}^{d-1}$ is a shelling of $\mathrm{pure}_d(\Gamma) \cup \mathrm{pure}_{d-1}(\Gamma)$. Hence $\mathrm{pure}_{d-2}(\mathrm{pure}_d(\Gamma) \cup \mathrm{pure}_{d-1}(\Gamma))$ is shellable by Lemma 2.4, thus has a shelling $\mathcal{T}^{d-2}$. Since this is a partial shelling of $\mathrm{pure}_{d-2}(\Gamma)$ and $\mathrm{pure}_{d-2}(\Gamma)$ is extendably shellable, $\mathcal{T}^{d-2}$ can be extended to a shelling $\mathcal{T}^{d-2}\mathcal{S}^{d-2}$ of $\mathrm{pure}_{d-2}(\Gamma)$. From this it is easily verified that $\mathcal{S}^d\mathcal{S}^{d-1}\mathcal{S}^{d-2}$ is a shelling of $\mathrm{pure}_d(\Gamma) \cup \mathrm{pure}_{d-1}(\Gamma) \cup \mathrm{pure}_{d-2}(\Gamma)$. After repeating this construction, a sequence $\mathcal{S}^d\mathcal{S}^{d-1}\cdots\mathcal{S}^0$ is constructed as a shelling of Γ. ∎

3 Extendable shellability of 2-dimensional hereditary-shellable complexes

For a given 2-dimensional simplicial complex, let $\mathcal{A}$ be some given subset of 2-faces of the simplicial complex. We denote by $\overline{\mathcal{A}}$ the rest of the 2-faces. We call a vertex an *$\mathcal{A}$-vertex* if it is contained in some 2-face of $\mathcal{A}$, and a *non-$\mathcal{A}$-vertex* otherwise. Similarly, we call an edge an *$\mathcal{A}$-edge* if it is contained in some 2-face of $\mathcal{A}$, and a *non-$\mathcal{A}$-edge* otherwise. Accordingly, we say a 2-face is an *$\mathcal{A}$-triangle* if it is in $\mathcal{A}$, and a *non-$\mathcal{A}$-triangle* if it is in $\overline{\mathcal{A}}$. We say a non-$\mathcal{A}$-triangle is of *type* (s, t) if it has s $\mathcal{A}$-vertices and t $\mathcal{A}$-edges. (Note that the possible types are (0-0), (1-0), (2-0), (2-1), (3-0), (3-1), (3-2), and (3-3).)

In what follows, we consider a pure 2-dimensional simplicial complex Γ that is not extendably shellable. Such Γ has a maximal partial shelling that is not complete. We let $\mathcal{F}$ be the set of the 2-faces (= facets) of such a partial shelling, and eventually we investigate the types of non-$\mathcal{F}$-triangles (= 2-faces in $\overline{\mathcal{F}}$, that is, 2-faces not in the partial shelling). Note here that neither $\mathcal{F}$ nor $\overline{\mathcal{F}}$ is empty. First, we immediately observe the following.

Lemma 3.1. *Assume that a pure 2-dimensional simplicial complex Γ is not extendably shellable, and let $\mathcal{F}$ be the set of the 2-faces of a maximal partial shelling that is not complete. Then each non-$\mathcal{F}$-triangle is of type (0-0), (1-0), (2-0), (3-0), or (3-1).*

Proof. If there is a non-$\mathcal{F}$-triangle F of a type not listed (i.e., type (2-1), (3-2), or (3-3)), then the maximal partial shelling can be extended by adding F to the last of the sequence, contradicting the maximality. ∎

For further investigation, we introduce an equivalence relation among the edge-path cycles (= closed walks), which describes the fundamental group of a simplicial complex in a combinatorial way. Here, an *edge-path* is a sequence v_0-v_1-v_2-$\cdots$-v_r of vertices, where every pair of consecutive vertices $\{v_i, v_{i+1}\}$ is an edge of the simplicial complex. An *edge-path cycle* is an edge-path such that $v_0 = v_r$. (We allow some intermediate vertices or edges to appear repeatedly in an edge-path or an edge-path cycle, i.e., we do not require the paths and cycles to be simple.)

Definition 3.2. We say two edge-path cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ are *simply equivalent* when
(R1) $\mathcal{C}_1 = u_0$-$\ldots$-u_k-u_{k+1}-$\ldots$-u_{r-1}-u_0, $\mathcal{C}_2 = u_0$-$\ldots$-u_k-v-u_{k+1}-$\ldots$-u_{r-1}-u_0, and $\{u_k, v, u_{k+1}\}$ is a 2-face of Γ, or
(R2) $\mathcal{C}_1 = u_0$-$\ldots$-u_k-$\ldots$-u_{r-1}-u_0 and $\mathcal{C}_2 = u_0$-$\ldots$-u_k-v-u_k-$\ldots$-u_{r-1}-u_0.
Two cycles $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if there is a sequence $\mathcal{C} = \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t = \mathcal{C}'$ such that $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are simply equivalent for each $1 \le i \le t-1$.

For a connected simplicial complex Γ, by fixing the base vertex v_0, the equivalence classes of the edge-path cycles form an edge-path fundamental group, and it is isomorphic to the usual fundamental group of the underlying space of Γ (e.g., [12, Theorem 3.7.3]). Especially, a connected simplicial complex Γ has only one equivalence class of edge-path cycles if and only if Γ is simply connected. Since a pure shellable complex of dimension ≥ 2 is simply connected (e.g., [1]), a pure shellable complex of dimension ≥ 2 has only one equivalence class of edge-path cycles. Especially, each cycle in a pure shellable complex of dimension ≥ 2 should be equivalent to a trivial cycle that consists of only one vertex.

We have the following lemma that restricts the structure of $\mathcal{A}$ and $\overline{\mathcal{A}}$ in a simply connected complex.

Lemma 3.3. *Let Γ be a simply connected pure 2-dimensional simplicial complex and $\mathcal{A}$ a subset of the facets of Γ. Assume there is an edge-path cycle consisting of $\mathcal{A}$-edges except for exactly one non-$\mathcal{A}$-edge. Then it cannot happen that all the non-$\mathcal{A}$-triangles are of type (3-1).*

Proof. Since every 2-face in $\overline{\mathcal{A}}$ is of type (3-1), it can be observed easily that the numbers of non-$\mathcal{A}$-edges in the two cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ in (R1) of Definition 3.2 have the same parity, regardless whether the 2-face $\{u_k, v, u_{k+1}\}$ is an $\mathcal{A}$-triangle or a non-$\mathcal{A}$-triangle. In (R2), too, it is obviously true that the two cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same parity of non-$\mathcal{A}$-edges. Thus, in Γ, in all equivalent edge-path cycles the numbers of non-$\mathcal{A}$-edges have the same parity. Hence, if there is an edge-path cycle consisting of $\mathcal{A}$-edges except exactly one non-$\mathcal{A}$-edge, then the cycle is not equivalent to the trivial cycle consisting of only one vertex that has no non-$\mathcal{A}$-edges. This contradicts the assumption that Γ is simply connected. ∎

This lemma plays a crucial role in our further discussion. Before that, we have the following corollary for the structure of maximal partial shellings.

Corollary 3.4. *Assume that a 2-dimensional pure simplicial complex Γ is shellable but not extendably shellable. Let $\mathcal{F}$ be the set of 2-faces of a maximal partial shelling that is not complete. Then it cannot happen that all the non-$\mathcal{F}$-triangles are of type (3-1).*

Proof. Let $\{a, b, c\}$ be a non-$\mathcal{F}$-triangle. This is of type (3-1) by assumption. So we assume, without loss of generality, that $\{b, c\}$ is an $\mathcal{F}$-edge, and $\{a, b\}$ and $\{a, c\}$ are non-$\mathcal{F}$-edges. Here, $\langle \mathcal{F} \rangle$ is a 2-dimensional shellable complex since $\mathcal{F}$ is a partial shelling, and thus the graph consisting of all the $\mathcal{F}$-vertices and $\mathcal{F}$-edges is connected by Lemma 2.2 and the remark after that. Hence, there exists a path consisting of $\mathcal{F}$-edges connecting a and c. On the other hand, Γ is simply connected since it is a pure 2-dimensional shellable complex. The statement follows from Lemma 3.3. ■

Now we prove our main theorem.

Theorem 3.5. *If a 2-dimensional simplicial complex Γ is hereditary-shellable, then* $\mathrm{pure}_2(\Gamma)$ *is extendably shellable.*

Proof. Assume there is a 2-dimensional hereditary-shellable simplicial complex Γ such that $\mathrm{pure}_2(\Gamma)$ is not extendably shellable. Here, $\mathrm{pure}_2(\Gamma)$ is shellable by Lemma 2.4, but not extendably shellable by assumption. Let $\mathcal{F}$ be the set of 2-faces of a maximal partial shelling of $\mathrm{pure}_2(\Gamma)$ that is not complete. Here, $\mathrm{pure}_2(\Gamma)$ is strongly connected by Lemma 2.2, and this implies that there is a non-$\mathcal{F}$-triangle $\{x, y, z\}$ that shares an $\mathcal{F}$-edge $\{x, y\}$ with an $\mathcal{F}$-triangle. Such a 2-face should be of type (3-1) by Lemma 3.1. So the two edges $\{x, z\}$ and $\{y, z\}$ are both non-$\mathcal{F}$-edges. For a non-$\mathcal{F}$-triangle of type (3-1), we say the $\mathcal{F}$-edge ($\{x, y\}$ of $\{x, y, z\}$) the *bottom edge*, and the vertex opposite to the $\mathcal{F}$-edge (z of $\{x, y, z\}$) the *apex vertex*. Since $\langle \mathcal{F} \rangle$ is 2-dimensional and shellable, the graph consisting of the $\mathcal{F}$-vertices and the $\mathcal{F}$-edges is connected by Lemma 2.2 and the remark after that, and thus there is a path consisting of $\mathcal{F}$-edges starting from the apex vertex z and ending at the bottom edge $\{x, y\}$.

Let $W \subseteq V$ be a set of vertices of minimum size such that $\Gamma[W]$ contains a non-$\mathcal{F}_W$-triangle F of type (3-1) together with a path of $\mathcal{F}_W$-edges connecting the apex vertex and the bottom edge of F, where $\mathcal{F}_W = \mathcal{F} \cap \Gamma[W]$ (i.e., the set of 2-faces of $\mathcal{F}$ contained in W). (Remark that the types of non-$\mathcal{F}_W$-triangles are considered with respect to $\mathcal{F}_W$, not $\mathcal{F}$.) Here, W is well-defined since $\Gamma = \Gamma[V]$ contains a non-$\mathcal{F}_V$-triangle of type (3-1) and a path of $\mathcal{F}_V$-edges connecting the apex vertex and the bottom edge, as is observed above. We note that, for each edge of the path of $\mathcal{F}_W$-edges, including the bottom edge of F, there exists at least one $\mathcal{F}_W$-triangle in $\Gamma[W]$ that contains the edge. (See Figure 1.) We say this path of $\mathcal{F}_W$-edges together with the incident $\mathcal{F}_W$-triangles as an *$\mathcal{F}_W$-triangle-path*, while we refer the path itself as the *path part* of the $\mathcal{F}_W$-triangle-path. Note that each vertex of W should be either on F, on the path of $\mathcal{F}_W$-edges connecting the apex vertex and the bottom edge, or on the $\mathcal{F}_W$-triangles that contain some $\mathcal{F}_W$-edge of the path. This follows from the minimality of W, since otherwise deleting the vertex does not corrupt the required property of W, contradicting the minimality. This implies that each vertex of W is incident to some $\mathcal{F}_W$-triangle of $\Gamma[W]$. Especially, we observe that the non-$\mathcal{F}_W$-triangles in $\Gamma[W]$ are of type (3-0) or (3-1) by Lemma 3.1. (This follows from that non-$\mathcal{F}_W$-triangles of type (3-2) or (3-3) are non-$\mathcal{F}$-triangles of type (3-2) or (3-3), since $\mathcal{F}_W \subseteq \mathcal{F}$.)

We show that $\Gamma[W]$ contains no non-$\mathcal{F}_W$-triangles of type (3-0) by means of contradiction. Assume $\{a, b, c\} \in \Gamma[W]$ is a non-$\mathcal{F}_W$-triangle of type (3-0). As discussed above, the minimality of W implies

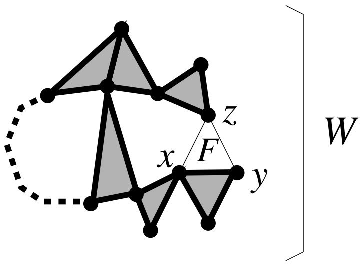

Figure 1: The set of vertices W. The gray 2-faces are $\mathcal{F}_W$-triangles, and the bold edges are $\mathcal{F}_W$-edges.

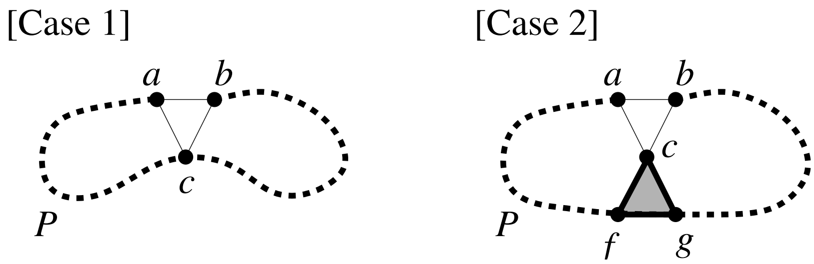

Figure 2: The $\mathcal{F}_W$-triangle-path P from a to b is not contained in $\mathrm{pure}_2(\Gamma[W \setminus \{c\}])$.

that each vertex of W is either on F, on the path of $\mathcal{F}_W$-edges connecting the apex vertex and the bottom edge, or on an $\mathcal{F}_W$-triangle incident to an edge on the path. First consequence we deduce from this fact is that each vertex of W is incident to some $\mathcal{F}_W$-triangle of $\Gamma[F]$. Especially this assures the existence of an $\mathcal{F}_W$-triangle $\{b, d, e\}$ in $\Gamma[W]$. (Here, the vertices a, b, c, d, e are all distinct.) Further, by Corollary 2.12, we can assume without loss of generality that there is a 2-face $\{a, b, d\}$. Note that this 2-face is a non-$\mathcal{F}_W$-triangle of type (3-1).

Second consequence we deduce from the fact is that $\Gamma[W]$ contains an $\mathcal{F}_W$-triangle-path P whose path part connects the apex vertex a and the bottom edge$\{b, d\}$ of the 2-face $\{a, b, d\}$ of type (3-1). But we observe that $\Gamma[W \setminus \{c\}]$ does not contain the $\mathcal{F}_W$-triangle-path P, since otherwise $W \setminus \{c\}$ contains a non-$\mathcal{F}_W$-triangle $\{a, b, d\}$ of type (3-1) with an $\mathcal{F}_W$-triangle-path P connecting a and $\{b, d\}$, contradicting the minimality of W. Such a situation can happen when c is contained in the path part of P (Case 1), or there is an edge $\{f, g\}$ on the path part of P such that there exists an $\mathcal{F}_W$-triangle $\{c, f, g\}$ in $\Gamma[W]$ and this is the only one $\mathcal{F}_W$-triangle in $\Gamma[W]$ that contains $\{f, g\}$ (Case 2), see Figure 2.

[Case 1] Assume c is on the path part of P. On the path part of P, there are $\mathcal{F}_W$-edges incident to c, and each of them are incident to an $\mathcal{F}_W$-triangle contained in $\Gamma[W]$. Let $\{c, f, g\}$ be one of such $\mathcal{F}_W$-triangles. (Note that a, b, c, f, g are all distinct.) By Corollary 2.12, we can without loss of generality assume there exists a non-$\mathcal{F}_W$-triangle $\{a, c, f\}$ of type (3-1) in $\Gamma[W]$ with $\{c, f\}$ on the path part of P. (By Corollary 2.12, there are four candidates for the 2-face to exist. In any case, the 2-face will be a non-$\mathcal{F}_W$-triangle of type (3-1) contained in $\Gamma[W]$. By exchanging the labels a with b, and f with g, if needed, we can assume $\{a, c, f\}$ exists, where at least one of $\{c, f\}$ or $\{c, g\}$ is on the path part of P. If $\{c,g\}$ is on the path part of P but $\{c,f\}$ is not, then we replace the path part of P by the simply equivalent path applying the rule (R1) of Definition 3.2 at the 2-face $\{c,f,g\}$. Then we have the situation that $\{a,c,f\}$ is an $\mathcal{F}_W$-triangle in $\Gamma[W]$ and $\{c,f\}$ is on the path part of P. We need not care in the following argument whether f appears before or after c on the path part of P starting from a. In Figure 3, the figures are written as f appearing before c, but we do not need to change the argument of the following proof in case f appears after c.) On the subpath $\overline{a\text{-}c}^P$ of the path part of P between a and c, we look for an edge $\{p,q\}$ nearest to a such that there is an $\mathcal{F}_W$-triangle $\{b,p,q\}$ or $\{c,p,q\}$ in $\Gamma[W]$.

If there exists no such $\{p,q\}$ at all, then $\Gamma[W \setminus \{b\}]$ contains an $\mathcal{F}_W$-triangle-path from the apex vertex a to the bottom edge $\{c,f\}$ of the non-$\mathcal{F}_W$-triangle $\{a,c,f\}$ of type (3-1). This contradicts the minimality of W.

If there exists an $\mathcal{F}_W$-triangle $\{c,p,q\}$ for the nearest edge $\{p,q\}$, then again $\Gamma[W \setminus \{b\}]$ contains an $\mathcal{F}_W$-triangle-path connecting the apex vertex a and the bottom edge $\{c,f\}$ of the non-$\mathcal{F}_W$-triangle $\{a,c,f\}$ of type (3-1), where the path part is the subpath $\overline{a\text{-}p}^P$ of P from a to p together with the edge $\{p,c\}$, contradicting the minimality of W. (Figure 3 (2).)

Consider the case that no $\mathcal{F}_W$-triangle $\{c,p,q\}$ exists for the nearest edge $\{p,q\}$ but an $\mathcal{F}$-triangle $\{b,p,q\}$ exists. By Corollary 2.12 and the existence of the two 2-faces $\{a,b,c\}$ and $\{b,p,q\}$, there exists one of the facets $\{a,b,p\}$, $\{a,b,q\}$, $\{b,c,p\}$, or $\{b,c,q\}$ of type (3-1). If $\{a,b,p\}$ (or $\{a,b,q\}$) exists, then $\Gamma[W \setminus \{c\}]$ contains an $\mathcal{F}_W$-triangle-path connecting the apex vertex and the bottom edge of the non-$\mathcal{F}_W$-triangle $\{a,b,p\}$ (resp. $\{a,b,q\}$) of type (3-1), where the path part is $\overline{a\text{-}p}^P$ (resp. $\overline{a\text{-}q}^P$) from a to p (resp. to q). This is a contradiction. (Figure 3 (3).) If $\{b,c,p\}$ (or $\{b,c,q\}$) exists, then look for an edge $\{r,s\}$ on the subpath $\overline{c\text{-}b}^P$ of P from c to b nearest to c such that an $\mathcal{F}_W$-triangle $\{a,r,s\}$ or $\{b,r,s\}$ exists in $\Gamma[W]$. If there exists no such edge $\{r,s\}$, then $\Gamma[W \setminus \{a\}]$ contains an $\mathcal{F}_W$-triangle-path connecting the apex vertex and the bottom edge of the non-$\mathcal{F}_W$-triangle $\{b,c,p\}$ (resp. $\{b,c,q\}$) of type (3-1), where the path part is the path $\overline{c\text{-}b}^P$. (Figure 3 (4).) If $\{b,r,s\}$ exists for the nearest $\{r,s\}$, then $\Gamma[W \setminus \{a\}]$ contains an $\mathcal{F}_W$-triangle path connecting the apex vertex and the bottom edge of the non-$\mathcal{F}_W$-triangle $\{b,c,p\}$ (resp. $\{b,c,q\}$) of type (3-1), where the path part is $\overline{c\text{-}r}^P$ together with the edge $\{r,b\}$. (Figure 3 (5).) If $\{b,r,s\}$ does not exist but $\{a,r,s\}$ exists for the nearest $\{r,s\}$, then $\Gamma[W \setminus \{b\}]$ contains an $\mathcal{F}_W$-triangle path connecting the apex vertex and the bottom edge of the non-$\mathcal{F}_W$-triangle $\{a,c,f\}$ of type (3-1), where the path part is $\overline{c\text{-}r}^P$ together with the edge $\{a,r\}$. (Figure 3 (6).) All these cases leads us to a contradiction.

[Case 2] By Corollary 2.12 and the existence of $\{a,b,c\}$ and $\{c,f,g\}$, we can assume without loss of generality that $\{a,c,f\}$ exists in $\Gamma[W]$. The rest of the argument is the same as Case 1 by replacing the edge $\{f,g\}$ of P by the edges $\{f,c\}$-$\{c,g\}$, and we are lead to a contradiction.

Thus we have shown that $\mathrm{pure}_2(\Gamma[W])$ has no non-$\mathcal{F}_W$-triangle of type (3-0).

On the other hand, $\mathrm{pure}_2(\Gamma[W])$ is a shellable pure 2-dimensional complex and thus is simply connected, and it has a cycle consisting of exactly one non-$\mathcal{F}_W$-edge $\{z,x\}$ and a path of $\mathcal{F}_{W_F}$-edges connecting z and x. Hence, by Lemma 3.3, it should contain some non-$\mathcal{F}_W$-triangles of type other than (3-1). Since all the vertices of W are $\mathcal{F}_W$-vertices, such non-$\mathcal{F}_W$-triangles should be of type (3-0) by Lemma 3.1. This is a contradiction.

■

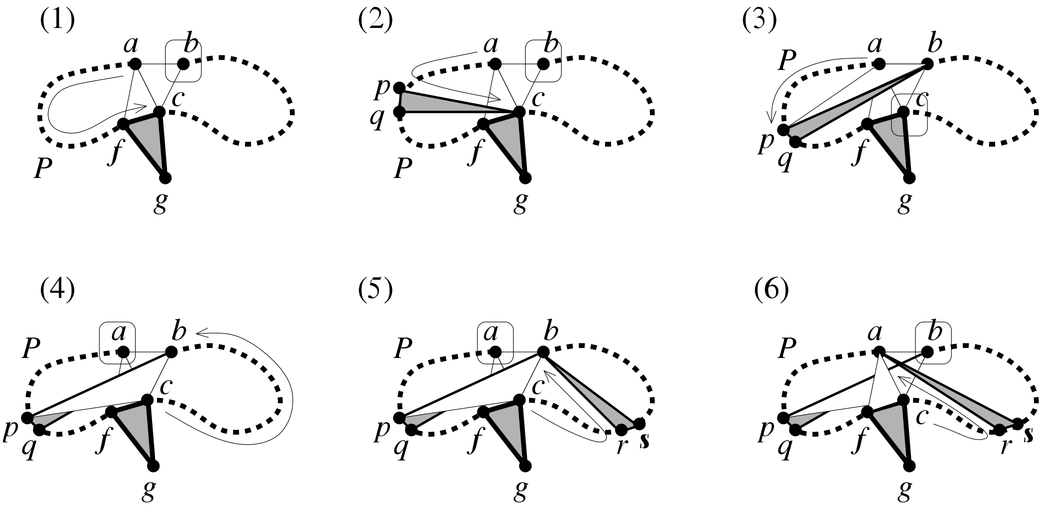

Figure 3: Several subcases for Case 1.

As a corollary, we immediately have the following.

Corollary 3.6. *If a pure* 2*-dimensional simplicial complex is hereditary-shellable, then it is extendably shellable.*

4 Matroid complexes and vertex decomposability

A *matroid complex* is a pure simplicial complex such that the restriction to any subset is pure. This definition of a matroid complex coincides with the simplicial complex formed by the independent sets of a matroid. A k-dimensional matroid complex corresponds to a matroid of rank $k+1$. See [2].

The following comes from the fact that every matroid complex is shellable ([2, 10]).

Proposition 4.1. *Every matroid complex is hereditary-shellable.*

By this, hereditary-shellable simplicial complexes can be seen as a kind of generalization of matroids. For matroid complexes, Björner and Eriksson [3] showed the following.

Theorem 4.2. *([3])* 2*-dimensional matroid complexes (i.e., matroid complexes of rank 3) are extendably shellable.*

Our Theorem 3.5 is a generalization of this theorem: Theorem 4.2 follows from Corollary 3.6 together with Proposition 4.1 and the fact that matroid complexes are pure.

On the other hand, Tracy Hall [11] showed the following.

Theorem 4.3. *([11]) There exists an 11-dimensional matroid complex that is not extendably shellable.*

(The example provided in [11] is a matroid complex consisting of the facets of a 12-dimensional crosspolytope.) This implies that our Theorem 3.5 and Corollary 3.6 can not be extended to general dimensions.

Remark. The generalization of Theorem 3.5 and Corollary 3.6 to higher dimensions is open even for the case of dimension 3. Especially, it is even unknown whether every 3-dimensional uniform matroid complex is extendably shellable or not. (Here, a *uniform matroid complex* of dimension k is a simplicial complex that has all the subsets of size $k+1$ of its vertex set as its facets. This corresponds to a uniform matroid of rank $k+1$.)

Vertex decomposability is another property that strengthens shellability. The definition is as follows.

Definition 4.4. ([5]) A simplicial complex Γ is *vertex decomposable* if either of the following holds.
(a) Γ has only one facet.
(b) There exists $x \in V$ such that $\mathrm{link}_\Gamma(\{x\})$ and $\Gamma[V \setminus \{x\}]$ are both vertex decomposable, and no facet of $\mathrm{link}_\Gamma(\{x\})$ is a facet of $\Gamma[V \setminus \{x\}]$.

Note that this definition is the generalized version that can be applied for nonpure complexes, proposed in [5]. Especially, when a simplicial complex Γ is restricted to be pure, the condition (b) of vertex decomposability is simplified as follows: (b') there exists a vertex x such that $\mathrm{link}_\Gamma(x)$ is vertex decomposable and $\Gamma[V \setminus \{x\}]$ is pure and vertex decomposable, which coincides with the classical definition of [10]. The vertex x in the condition (b) is called a *shedding vertex*.

As shown in [5, Theorem 11.3], vertex decomposable simplicial complexes are shellable. On the other hand, for matroid complexes the following is well-known.

Theorem 4.5. *([10]) Matroid complexes are vertex decomposable.*

In view of the fact that Theorem 4.2 is strengthened into Corollary 3.6, it is natural to ask whether Theorem 4.5 can be strengthened for pure hereditary-shellable complexes or not. But, in the following example, we see that this is not possible.

Example 4.6. We give an example of a pure 2-dimensional simplicial complex that is hereditary-shellable but not vertex decomposable. Our example is a pure 2-dimensional simplicial complex such that the deletion by arbitrary one vertex is non-pure. For such a simplicial complex, no vertex can be a shedding vertex and thus the complex can not be vertex decomposable.

Prepare three partite sets $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4\}$, and $C = \{c_1, c_2, c_3, c_4\}$, and define a simplicial complex Γ on the set of twelve vertices $A \cup B \cup C$ with the following facets:

- $\{a_i, b_j, c_k\}(1 \leq i, j, k \leq 4)$, which we call the matroidal part, and
- the following twelve extra facets:

$$\begin{gathered}\{a_2, b_1, b_2\}, \{a_2, a_3, b_2\}, \{a_3, b_2, b_3\}, \{a_3, a_4, b_3\},\\ \{b_1, c_1, c_2\}, \{b_1, b_4, c_2\}, \{b_4, c_2, c_3\}, \{b_4, b_2, c_3\},\\ \{c_1, a_3, a_1\}, \{c_1, c_4, a_1\}, \{c_4, a_1, a_4\}, \{c_4, c_2, a_4\}.\end{gathered}$$

It is easy to verify that the facets of the matroidal part form a matroid complex. The twelve extra facets are depicted in Figure 4, where each facet is expressed by a triangle. In the figure, the vertices with the same label are identified. In the following we observe this simplicial complex Γ satisfies the required properties.

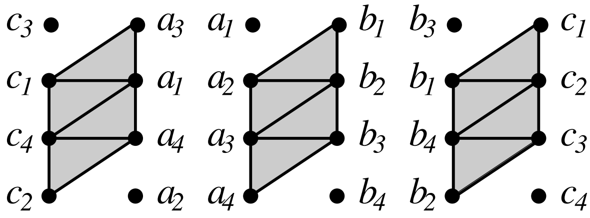

Figure 4: A simplicial complex that is hereditary-shellable but not vertex decomposable

First, shellability of this simplicial complex Γ is verified as follows. The matroidal part of the complex before adding extra facets is a matroid complex, so we can make a partial shelling consisting of all the facets of the matroidal part. As for the twelve extra facets, each facet shares two edges with the matroidal part. Therefore we can extend the partial shelling to the entire facets.

When a subset $U \subseteq A \cup B \cup C$ intersects with all the three partite sets A, B, and C, the shellability of the restriction $\Gamma[U]$ is shown in the same way. By such a subset U, the matroidal part is restricted to a matroid complex of dimension 2, thus $\Gamma[U]$ has a partial shelling on this part. Each additional facet either remains a 2-dimensional facet with two edges sharing with the matroidal part, or is restricted to a face of smaller dimension that is not a facet. So it is easily verified that the partial shelling on the matroidal part can be extended to the whole $\Gamma[U]$.

What remains is the case of $\Gamma[U]$ such that U is a subset of one of the sets $A \cup B$, $B \cup C$, or $A \cup C$. But we immediately see $\Gamma[U]$ is shellable for this case because each of the restrictions $\Gamma[A \cup B]$, $\Gamma[B \cup C]$, and $\Gamma[A \cup C]$ are isomorphic to Γ_3 of Example 2.8 that is hereditary-shellable. Therefore any restriction to a subset is shellable, and hence Γ is hereditary-shellable.

On the other hand, the deletion of any one vertex is not pure, since for each vertex x there is a facet $\{x, y, z\}$ among the twelve extra facets such that two vertices y and z belong to the same partite set different from the one x belongs to, while the facet $\{x, y, z\}$ is the only one facet that contains the edge $\{y, z\}$. (For example, for the vertex a_1, there is a facet $\{c_1, c_4, a_1\}$. After deleting a_1, the edge $\{c_1, c_4\}$ will become a facet.) Therefore, in Γ there exists no vertex x that satisfies the condition (b') (nor (b)), and hence we conclude that Γ is not vertex decomposable.

The existence of this 2-dimensional example Γ shows that there are d-dimensional pure simplicial complexes with the same property for all $d \geq 2$: for new vertices $v_1, v_2, \ldots, v_{d-2}$ outside of $A \cup B \cup C$, $v_1 * v_2 * \cdots * v_{d-2} * \Gamma = \{H_1 \cup H_2 : H_1 \subseteq \{v_1, v_2, \ldots, v_{d-2}\}, H_2 \in \Gamma\}$, i.e., the $(d-2)$-fold cone over Γ, is a d-dimensional pure simplicial complex that is hereditary-shellable but not vertex decomposable, since both hereditary-shellability and vertex decomposability are preserved under the operation of taking cones.

We note that there are simplicial complexes that are vertex decomposable but not hereditary-shellable: $\langle \{a, b\}, \{b, c\}, \{c, d\}, \{d, e\} \rangle$ for example.

Remark. In [9, Sec. 3], it is shown that, in the class of simplicial complexes of dimensions ≤ 2 and that of flag complexes, the properties lying between hereditary-shellability and hereditary-sequential Cohen-Macaulayness, or hereditary-partitionability, are all equivalent. On the other hand, our Example 4.6 shows that hereditary-shellability and hereditary-vertex decomposability are different in dimensions

≥ 2. (In the class of flag complexes, the result of Woodroofe [14] implies that hereditary-shellability and hereditary-vertex decomposability are equivalent, see also [9, Sec. 3].)

5 Pure skeletons of strong obstructions to shellability

The concept of obstructions to shellability is introduced by Wachs in [13].

Definition 5.1. A simplicial complex Γ on V is an *obstruction to shellability* if Γ is nonshellable and $\Gamma[U]$ is shellable for every $U \subsetneq V$.

The difference between a hereditary-shellable complex and an obstruction to shellability is whether the complex itself is shellable or nonshellable. There is a natural relation between these two concepts that a simplicial complex is hereditary-shellable if and only if it has no obstructions to shellability as its restrictions, see [9, Section 3].

In [9, Section 4], the following strengthened concept is introduced.

Definition 5.2. A simplicial complex Γ on V is a *strong obstruction to shellability* if Γ is nonshellable and $\mathrm{link}_{\Gamma[U]}(A)$ is shellable for any $U \subseteq V$ and $A \in \Gamma$ unless $W = U$ and $A = \emptyset$ (i.e., unless $\mathrm{link}_{\Gamma[U]}(A) = \Gamma$).

In other words, obstructions to shellability are minimal nonshellable complexes with respect to restrictions, while strong obstructions to shellability are minimal nonshellable complexes with respect to restrictions and links. The same concept is discussed in [15] under the name "dc-obstructions to shellability". Strong obstructions to shellability can be used, instead of obstructions to shellability, for a characterization of hereditary-shellability as follows.

Proposition 5.3. *([9, Prop. 4.3]) A simplicial complex Γ on V is hereditary-shellable if and only if $\mathrm{link}_{\Gamma[U]}(A)$ is not a strong obstruction to shellability for any $U \subseteq V$ and $A \in \Gamma[U]$.*

As is shown in [9, Prop. 4.2], strong obstructions to shellability are characterized by a weaker condition as follows.

Proposition 5.4. *([9, Prop 4.2]) A simplicial complex Γ on V is a strong obstruction to shellability if and only if Γ is nonshellable, $\Gamma[U]$ is shellable for any $U \subsetneq V$, and $\mathrm{link}_\Gamma(A)$ is shellable for any $A \in \Gamma$ with $A \neq \emptyset$.*

All the obstructions to shellability (and thus all the strong obstructions to shellability) are determined for dimensions up to 2 in [9], but not many are known for dimensions ≥ 3. In this section, we develop a discussion of the structure of strong obstructions to shellability using Theorem 3.5.

Lemma 5.5. *For a strong obstruction to shellability Γ on V of dimension $k \geq 3$, there is no subset $U \subseteq V$ such that $\mathrm{pure}_2(\mathrm{pure}_2(\Gamma)[U])$ $(= \mathrm{pure}_2(\Gamma[U]))$ is the pure 2-skeleton of some 2-dimensional obstruction to shellability.*

Proof. Let $\Delta = \mathrm{pure}_2(\Gamma) \cup \{\{x, y\} : x, y \in V\}$. First, we show that Δ is hereditary-shellable.

Assume $U \subsetneq V$. Here, we have that $\mathrm{pure}_2(\Delta[U]) = \mathrm{pure}_2(\Gamma[U])$ is shellable since $\Gamma[U]$ is shellable and by Lemma 2.4. We also have that $\mathrm{pure}_1(\Delta[U])$ is connected. Hence $\Delta[U]$ is shellable by Lemma 2.5.

Let $U = V$. If Δ is not shellable, then $\Delta[U] = \Delta$ is a 2-dimensional obstruction to shellability. Further, for any vertex $v \in \Delta$,

$$\begin{aligned}\mathrm{link}_\Delta(\{v\}) &= \mathrm{link}_{(\mathrm{pure}_2(\Gamma)\cup\{\{x,y\}:x,y\in V\setminus\{v\}\})}(\{v\}) = \mathrm{link}_{\mathrm{pure}_2(\Gamma)}(\{v\}) \cup (V \setminus \{v\}) \\ &= \mathrm{pure}_1(\mathrm{link}_\Gamma(\{v\})) \cup (V \setminus \{v\}).\end{aligned}$$

Since Γ is a strong obstruction to shellability, $\mathrm{pure}_1(\mathrm{link}_\Gamma(\{v\}))$ is shellable by Lemma 2.4, and thus $\mathrm{link}_\Delta(\{v\})$ is also shellable. For $A \in \Delta$ with $\dim A > 0$, it is clear that $\mathrm{link}_\Delta(A)$ is shellable. Hence Δ is a 2-dimensional strong obstruction to shellability. Thus we have that Γ is a k-dimensional simplicial complex derived by adding some faces of dimensions from 3 to k to a 2-dimensional obstruction to shellability Δ and removing some edges. But, by checking the list of 2-dimensional strong obstructions to shellability given in [9, Theorem 2.11], there is no 2-dimensional strong obstruction to shellability that has places where 3- and higher dimensional faces can be added (i.e., in any 2-dimensional strong obstructions, there is no subcomplex that forms the 2-dimensional skeleton of a higher dimensional simplex), a contradiction. Hence we conclude that, for $U = V$, $\Delta[U] = \Delta$ is shellable. Therefore, we have shown that $\Delta[U]$ is shellable for any $U \subseteq V$, hence is hereditary-shellable, as intended.

Now assume that $\mathrm{pure}_2(\mathrm{pure}_2(\Gamma)[U])$ is the pure 2-skeleton of some 2-dimensional obstruction to shellability for $U \subseteq V$. This implies that there exists a 2-dimensional obstruction to shellability Π that is derived by adding some edges to $\mathrm{pure}_2(\Gamma)[U]$. Since the simplicial complex derived by adding edges to a 2-dimensional obstruction to shellability is again an obstruction to shellability ([9, Prop. 2.10]), $\Pi \cup \{\{x,y\} : x,y \in V\} = \mathrm{pure}_2(\mathrm{pure}_2(\Gamma)[U]) \cup \{\{x,y\} : x,y \in V\} = \Delta[U]$ is an obstruction to shellability, a contradiction.

■

Corollary 5.6. *If* Γ *is a strong obstruction to shellability of dimension* $k \geq 3$*, then* $\mathrm{pure}_2(\Gamma)$ *is shellable.*

Proof. Assume $\mathrm{pure}_2(\Gamma)$ is nonshellable. Let $\Delta = \mathrm{pure}_2(\Gamma) \cup \{\{x,y\} : x,y \in V\}$. Then Δ is nonshellable by Lemma 2.4 since $\mathrm{pure}_2(\Delta) = \mathrm{pure}_2(\Gamma)$, and thus there exists $U \subseteq V$ with $U \neq \emptyset$ such that $\Delta[U]$ is an obstruction to shellability. Here, $\dim \Delta[U] = 2$ since there is no obstruction to shellability of dimension 0, and the only one 1-dimensional obstruction to shellability is $\langle \{a,b\}, \{c,d\} \rangle$ (see [13, Prop. 2]) that does not appear as $\Delta[U]$. Hence, $\mathrm{pure}_2(\Delta[U]) = \mathrm{pure}_2(\mathrm{pure}_2(\Gamma)[U])$ is the pure 2-skeleton of an obstruction to shellability, which contradicts Lemma 5.5. ■

The following theorem is an application of Theorem 3.5.

Theorem 5.7. *For a strong obstruction to shellability* Γ *of dimension* $k \geq 3$*,* $\mathrm{pure}_2(\Gamma)$ *is extendably shellable.*

Proof. Let $\Delta = \mathrm{pure}_2(\Gamma) \cup \{\{x,y\} : x,y \in V\}$. Then Δ is hereditary-shellable by the argument in the proof of Corollary 5.6. By Theorem 3.5, $\mathrm{pure}_2(\Delta) = \mathrm{pure}_2(\Gamma)$ is extendably shellable. ■

As an application of this Theorem 5.7, we have the following theorem that describes the structure of pure skeletons of 3-dimensional strong obstructions to shellability, which is the goal of our discussion of this section.

Theorem 5.8. *For a* 3*-dimensional strong obstruction to shellability* Γ*,* $\mathrm{pure}_3(\Gamma)$ *is nonshellable and* $\mathrm{pure}_i(\Gamma)$ *is shellable for* $i \leq 2$*.*

Proof. $\mathrm{pure}_0(\Gamma)$ is trivially extendably shellable, and $\mathrm{pure}_1(\Gamma)$ is extendably shellable since $\mathrm{pure}_1(\Gamma)$ is connected (see [9, Lemma 2.8]). $\mathrm{pure}_2(\Gamma)$ is also extendably shellable by Theorem 5.7. If $\mathrm{pure}_3(\Gamma)$ is shellable, then Γ becomes shellable by Proposition 2.14, a contradiction. Therefore, $\mathrm{pure}_3(\Gamma)$ is nonshellable. ■

The same statement holds for 2-dimensional (strong) obstructions to shellability by the list of [9, Theorem 2.12]: for a 2-dimensional obstruction to shellability Γ, $\mathrm{pure}_2(\Gamma)$ is nonshellable while $\mathrm{pure}_i(\Gamma)$ is shellable for $0 \leq i \leq 1$. Note that the statement of Theorem 5.8 is not true for non-strong obstructions to shellability: $\{\{a, b, c, d\}, \{a, b, e\}, \{c, d, e\}\}$ is a 3-dimensional obstruction to shellability of which the pure 3-skeleton is shellable and the pure 2-skeleton is nonshellable.

One of the open questions is whether obstructions to shellability and obstructions to sequential Cohen-Macaulayness coincide or not ([9, Question 4.5]). For this question, the answer is positive for dimensions ≤ 2 ([9, Theorem 3.12]), but open for dimensions ≥ 3. The following may contribute for attacking this question for dimension 3.

Corollary 5.9. *Obstructions to shellability coincide with obstructions to sequential Cohen-Macaulayness for dimension* 3 *if* $\mathrm{pure}_3(\Gamma)$ *is not Cohen-Macaulay for all strong obstructions to shellability* Γ *of dimension* 3*.*

Proof. Obstructions to shellability coincide with obstructions to sequential Cohen-Macaulayness for dimension 3 if all 3-dimensional strong obstructions to shellability are not Cohen-Macaulay by [9, Theorem 4.7] with [9, Theorem 3.10]. For a 3-dimensional strong obstruction to shellability Γ and $0 \leq i \leq 2$, $\mathrm{pure}_i(\Gamma)$ is shellable, hence Cohen-Macaulay, by Theorem 5.8. Thus Γ is sequentially Cohen-Macaulay if and only if $\mathrm{pure}_3(\Gamma)$ is Cohen-Macaulay. (A simplicial complex Δ is sequentially Cohen-Macaulay if and only if $\mathrm{pure}_i(\Delta)$ is Cohen-Macaulay for all $0 \leq i \leq \dim \Delta$. Note that pure shellable complexes are Cohen-Macaulay, see [6, 7].) ■

By the corollary above, it is important to know what kind of 3-dimensional pure simplicial complexes can appear as the pure 3-skeletons of strong obstructions to shellability. The following corollary may be of some help for this line of study.

Corollary 5.10. *For a* 3*-dimensional strong obstruction to shellability* Γ*,* $\widetilde{\Gamma} = \Gamma \cup \{\{x, y, z\} : x, y, z \in V\}$ *is also a* 3*-dimensional strong obstruction to shellability.*

Proof. For $U = V$, $\widetilde{\Gamma}[U] = \widetilde{\Gamma}$ is nonshellable by Lemma 2.4 together with the fact that $\mathrm{pure}_3(\Gamma)$ is nonshellable by Theorem 5.8.

Consider the case $U \subsetneq V$. Then $\mathrm{pure}_3(\widetilde{\Gamma}[U])$ is shellable since $\mathrm{pure}_3(\widetilde{\Gamma}[U]) = \mathrm{pure}_3(\Gamma[U])$ and $\Gamma[U]$ is shellable. Also we have that $\mathrm{pure}_2(\widetilde{\Gamma}[U])$ is extendably shellable by Theorem 4.2 since it is a 2-dimensional uniform matroid complex, and that $\mathrm{pure}_1(\widetilde{\Gamma}[U])$ is extendably shellable since it is connected. Hence $\widetilde{\Gamma}[U]$ is shellable by Proposition 2.14. (In the proof, when $\mathrm{pure}_i(\widetilde{\Gamma}[U])$ is empty, we just assume it is extendably shellable and the proof works as well.)

The fact that $\mathrm{link}_{\widetilde{\Gamma}}(A)$ is shellable for every $A \in \widetilde{\Gamma}$ with $A \neq \emptyset$ is clear.

■

Acknowledgment

The first author was partly supported by KAKENHI #25400191, #25350434, and #21740062.

References

[1] A. Björner: Topological methods. Handbook of Combinatorics, Vol. II, 1819–1872, Elsevier, Amsterdam, 1995.

[2] A. Björner, The homology and shellability of matroids and geometric lattices, in Matroid Applications, N. White ed., Cambridge Univ. Press, 1992, 226-283.

[3] A. Björner and K. Eriksson, Extendable shellability for rank 3 matroid complexes, Discrete Math. 132 (1994), 373-376.

[4] A. Björner and M. Wachs, Shellable non-pure complexes and posets I, Trans. Amer. Math. Soc. 348 (1996) 1299–1327.

[5] A. Björner and M. Wachs, Shellable non-pure complexes and posets II, Trans. Amer. Math. Soc. 349 (1997) 3945–3975.

[6] A. Björner, M.L. Wachs and V. Welker, On sequentially Cohen-Macaulay complexes and posets, Israel J. Math. 169 (2009), 295-316.

[7] A.M. Duval, Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes, Electron. J. Combin. 3(1) (1996), R21.

[8] G. Danaraj and V. Klee, Which spheres are shellable?, Annals of Discrete Mathematics 2 (1978), 33-52.

[9] M. Hachimori and K. Kashiwabara, Obstructions to shellability, partitionability, and sequential Cohen-Macaulayness, J. Combin. Theory, Ser. A **108** (2011), 1608-1623.

[10] J.S. Provan and L.J. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra, Math. Oper. Res. 5 (1980), 576–594.

[11] H. Tracy Hall, Counterexamples in discrete geometry, Ph.D. thesis, UC Berkeley, 2004, 61 pages.

[12] E. H. Spanier, Algebraic topology, McGraw-Hill, New York, 1966.

[13] M. Wachs, Obstructions to shellability, Discrete Computat. Geom., 22 (1999),95–103.

[14] R. Woodroofe, Vertex decomposable graphs and obstructions to shellability, Proc. Amer. Math. Soc. 137 (2009), 3235-3246.

[15] R. Woodroofe, Chordal and sequentially Cohen-Macaulay clutters, Electron. J. Combin. **18** (2011), no. 1, Paper 208.